A finite volume scheme with variable Péclet number for a nonlinear convection-diffusion equation arising in petroleum engineering

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February 28, 2005

Abstract

This paper presents a new finite volume scheme designed for the approximation of a nonlinear convection-diffusion equation arising in petroleum engineering. The convection part of the flux is written as a linear combination between an upwind scheme and a centered scheme. The parameter of the combination is computed according to the diffusion term in order to make the scheme stable and to reduce numerical diffusion. This scheme satisfies good mathematical properties and is shown to be convergent assuming that the total throughput is a given C^1-function. In practice, this scheme is easy to implement and can be used in a time explicit or implicit form, which enables the use of large time steps during the simulations.
Keywords: Flows in porous media, Nonlinear parabolic equation, Finite volume methods.

1 Introduction

We consider a two-phase flow through a porous medium Ω, for example an oil-water flow in a reservoir or in a sedimentary basin. Both phases are supposed to be immiscible, incompressible with constant viscosity and composed of only one component. We denote by T the duration of the flow. Taking into account the pressure gradient, the gravity and the capillary effects, the generalized Darcy's law (see Aziz and Settari [1979], Bear [1972], Peaceman [1977]) states that the saturation $u : \Omega \times (0,T) \to \mathbb{R}$ and the pressure $p : \Omega \times (0,T) \to \mathbb{R}$ are solutions to the following system:

$$\begin{cases} \phi \dfrac{\partial u}{\partial t} + \operatorname{div}\Big(\mathcal{K}\eta_1(u)\Big(\rho_1 g \nabla z - \nabla(p + \pi(u))\Big)\Big) = 0, \\ -\phi \dfrac{\partial u}{\partial t} + \operatorname{div}\Big(\mathcal{K}\eta_2(u)(\rho_2 g \nabla z - \nabla p)\Big) = 0 \end{cases} \tag{1.1}$$

where ϕ stands for the porosity of the medium, $\mathcal{K}$ is the absolute permeability of the rock, the subscript 1 represents the nonwetting phase and the subscript 2 the wetting phase, u is the saturation of the nonwetting phase, p is the pressure of the wetting phase, ρ_α is the density of the phase α, $\alpha \in \{1,2\}$, g is the gravity acceleration, $\eta_\alpha(u)$ is the mobility of the phase α, $\pi(u)$ is the capillary pressure.
We assume that the boundary of the domain is impermeable, i.e.,

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$$\begin{cases} \mathcal{K}\eta_2(u)\Big(\rho_2 g\nabla z - \nabla p\Big).\mathbf{n} = 0, \\ \mathcal{K}\eta_1(u)\Big(\rho_1 g\nabla z - \nabla(p+\pi(u))\Big).\mathbf{n} = 0 \end{cases}$$

where $\mathbf{n}$ denotes the unit normal outward to $\partial\Omega$. With such boundary conditions, the pressure field is known up to an additive constant.
The initial values of the saturation are given by $u(.,0) = u_{\text{ini}}(.)$.
Introducing the global pressure $\bar{p}$ (see Chavent and Jaffré [1986]) defined by $\bar{p} = p + \int_0^u \frac{\eta_1}{\eta_T}(v)\pi^{'}(v)dv$, the system (1.1) may be reformulated (see Michel [2003]) as

$$\begin{cases} \operatorname{div}(\mathbf{Q}) = 0, \\ \phi\dfrac{\partial u}{\partial t} + \operatorname{div}\Big(f(u,\mathbf{Q},\mathbf{G}) - \mathcal{K}\nabla\varphi(u)\Big) = 0, \end{cases} \tag{1.2}$$

where $\mathbf{Q}$ is the total flux defined by

$$\mathbf{Q} = \mathcal{K}\Bigg(\Big(\eta_1(u)\rho_1 + \eta_2(u)\rho_2\Big)g\nabla z - \eta_T(u)\nabla\bar{p}\Bigg), \tag{1.3}$$

with $f(u,\mathbf{Q},\mathbf{G}) = \frac{\eta_1}{\eta_T}(u)\mathbf{Q} + \frac{\eta_1\eta_2}{\eta_T}(u)\mathbf{G}$, $\mathbf{G} = \mathcal{K}(\rho_1 - \rho_2)g\nabla z$, $\varphi^{'}(u) = \frac{\eta_1(u)\eta_2(u)}{\eta_1(u)+\eta_2(u)}\pi^{'}(u)$.
Throughout this paper, the following hypotheses are taken for granted.

Assumptions 1.1

A1-1. *In the general case we can consider Ω as an open polygonal bounded connected subset of $\mathbb{R}^d$ (in practice $d = 1, 2$ or 3) and T as a positive given constant. But to simplify our study, we assume that Ω is a rectangle for $d = 2$ or a parallelepiped for $d = 3$.*

A1-2. $\phi, \mathcal{K} \in L^\infty(\Omega)$ *with* $0 < \phi(x) < 1$ *and* $0 < C_{\mathcal{K},inf} \leq \mathcal{K}(x) \leq C_{\mathcal{K},sup}$ *for a.e.* $x \in \Omega$.

A1-3. *We assume that, for all $\alpha \in \{1,2\}$, $\eta_\alpha : \mathbb{R} \to \mathbb{R}^+$ is a Lipschitz continuous function. We denote by C_α its Lipschitz constant. The function η_1 is strictly increasing on $(0,1)$, $\eta_1(u) = 0$ for all $u \leq 0$ and $\eta_1(u) = \eta_1(1)$ for all $u \geq 1$.*

Conversely the function η_2 is strictly decreasing on $(0,1)$, $\eta_2(u) = \eta_2(0)$ for all $u \leq 0$ and $\eta_2(u) = 0$ for all $u \geq 1$.

Moreover we assume that the total mobility $\eta_T = \eta_1 + \eta_2$ is bounded away from 0, i.e. there exists $\beta > 0$ such that $\beta = \inf_{u\in\mathbb{R}} \eta_T(u)$. We denote $\gamma = \sup_{u\in\mathbb{R}} \eta_T(u)$.

A1-4. *The capillary pressure is a $C^1(\mathbb{R},\mathbb{R})$-function which is strictly increasing on $(0,1)$.*

A1-5. $u_{\text{ini}} \in L^\infty(\Omega)$ *and* $0 \leq u_{\text{ini}}(x) \leq 1$ *for a.e.* x *on* Ω.

A1-6. *For all $\alpha \in \{1,2\}$ the densities ρ_α are constant and $\rho_1 < \rho_2$.*

Remark 1: Under Assumptions A1-3 and A1-4, φ is a $C^1(\mathbb{R},\mathbb{R})$-function which is Lipschitz continuous and strictly increasing on $(0,1)$. We denote by L_φ its Lipschitz constant. □

The existence, the uniqueness and the regularity of weak solutions to such problems have been studied in Alt and di Benedetto [1985], Antontsev et al. [1990], Chavent and Jaffré [1986], Chen [2001], Chen and Ewing [1999], Feng [1995], Langlo and Espedal [1992], Gagneux and Madaune-Tort [1996], Kroener and Luckhaus [1984] under various assumptions.

Here we are concerned with a finite volume approximation for (1.2). Without gravity we find in Michel [2003] a cell-centered finite volume scheme for (1.2). It consists in a centered finite difference scheme for the first equation and an upwind weighting scheme for the convection term $f(u, \mathbf{Q}, 0)$ coupled with a finite difference scheme for the gradient $\nabla\varphi(u)$. This scheme satisfies estimates in pressure and saturation and converges under the assumption that the parabolic term is not strongly degenerate.

In this paper we study a new finite volume scheme which relies on the use of a variable Péclet number to discretize (1.2). This scheme is designed to use the nonlinear diffusion term $\varphi(u)$ in order to take an approximation as centered as possible for the convection term $f(u, \mathbf{Q}, \mathbf{G})$, which reduces numerical diffusion. Slope limiters methods (see Brenier and Jaffré [1991]) can also reduce numerical diffusion but, in practice, they are limited to a time explicit discretization of the saturations in the fluxes. Here this scheme can be used in a time explicit or implicit form. In the latter form, the scheme is unconditionally stable and so large time steps can be used during the simulations. An other advantage of this scheme is the simplicity of its implementation.
In this paper we only detail the implicit case but all results established in the following are satisfied by the explicit scheme. First we recall classical pressure estimates. Then we prove the L^∞-stability of the saturation calculation (Proposition 2.2) and the existence of discrete solutions (Proposition 2.3) in pressure and saturation. The convergence of the saturation scheme is obtained assuming that the total throughput is a given $C^1(\Omega \times (0,T))$-function (Theorem 3.1). The last part is devoted to numerical tests (§4) where both forms of the scheme, the implicit and the explicit forms, are used.

2 A finite volume scheme for the coupled system

We briefly recall the definition of an admissible discretization of $\Omega \times (0,T)$ for the cell-centered finite volume method. Complete and detailed assumptions can be found in Eymard et al. [2000].

2.1 Admissible discretization of $\Omega \times (0,T)$

Definition 2.1 (Admissible mesh of Ω) *An admissible finite volume mesh of Ω, denoted by $\mathcal{M}$, is composed of a triplet $(\mathcal{T}, \mathcal{E}, \mathcal{P})$.*

- *$\mathcal{T}$ is a set of volumes K whose closure covers $\bar{\Omega}$. We denote by $\partial K = \bar{K} \setminus K$ the boundary of a volume K and by $m(K)$ its measure.*
- *$\mathcal{E}$ is the set of all edges, $\mathcal{E}_{int}$ the set of inner edges, $\mathcal{E}_{ext}$ the set of boundary edges, $\mathcal{E}_K$ the set of the edges of a volume K. An edge σ such that $\bar{\sigma} = \partial K \cap \partial L$ is also denoted by $K|L$. The set of the neighbouring volumes of a volume K is represented by $N(K) = \{L \in \mathcal{T},\, \sigma = K|L \in \mathcal{E}_K\}$. For all $\sigma \in \mathcal{E}$, we denote by $\mathbf{n}_{K,L}$ (resp. $\mathbf{n}_\sigma$) the unit normal of σ outward to K for $\sigma = K|L$ (resp. for $\sigma \in \mathcal{E}_{ext}$) and by $m(\sigma)$ its measure.*
- *$\mathcal{P}$ refers to a family of points $(x_K)_{K\in\mathcal{T}}$ where $\forall K \in \mathcal{T}$, $x_K \in K$ and where, for all $L \in N(K)$, the straight line going through x_K and x_L is orthogonal to $K|L$. For $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, we denote by $d_{K,\sigma}$ the distance between x_K and σ. If $\sigma = K|L$, we set $d_{K|L}$ the distance between x_K and x_L and $\tau_{K|L} = \dfrac{m(K|L)}{d_{K|L}}$ the transmissivity through $K|L$. If $\sigma \in \mathcal{E}_{ext}$, the transmissivity τ_σ through σ is given by $\tau_\sigma = \dfrac{m(\sigma)}{d_{K,\sigma}}$.*

We set $\mathrm{size}(\mathcal{T}) = \sup\{diam(K), K \in \mathcal{T}\}$ *and* $\mathrm{regul}(\mathcal{M}) = \max\limits_{K\in\mathcal{T},\sigma\in\mathcal{E}_K} \left(\dfrac{\mathrm{diam}(K)}{d_{K,\sigma}}\right)$.

In this paper, for the sake of simplicity, we restrict our study to constant time steps. But all results stated in the following can be adjusted to variable time steps.

Definition 2.2 (Admissible discretization of $\Omega \times (0,T)$) *An admissible discretization $\mathcal{D}$ of $\Omega \times (0,T)$ is composed of a pair $(\mathcal{M}, M)$ where $\mathcal{M}$ is an admissible discretization of Ω and $M \in \mathbb{N}^*$. We denote $\delta t = \frac{T}{M}$ and $t_n = n\delta t$. We denote* $\mathrm{size}(\mathcal{D}) = \max(\mathrm{size}(\mathcal{M}), \delta t)$.

Now let us define some notations. For a variable u we denote by u_K^{n+1} its approximation over the volume K and the time interval $(n\delta t, (n+1)\delta t]$ and by u_K^0 the piecewise constant approximation of the initial condition. We denote by

- $\mathcal{X}(\mathcal{T})$ the set of piecewise constant functions over the mesh $\mathcal{T}$: $u_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ is defined, for all $x \in \Omega$, by $u_{\mathcal{T}}(x) = u_K$ if $x \in K$,
- $\mathcal{X}(\mathcal{D})$ the set of piecewise constant functions over the discretization $\mathcal{D}$: $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ is defined for all $n \in \{0 \dots M\}$ and for all $t \in (n\delta t, (n+1)\delta t]$ by $u_{\mathcal{D}}(.,t) = u_{\mathcal{T}}^{n+1} \in \mathcal{X}(\mathcal{T})$ and by $u_{\mathcal{D}}(.,0) = u_{\mathcal{T}}^0 \in \mathcal{X}(\mathcal{T})$.

2.2 Definition of the scheme

Let $\mathcal{D}$ be an admissible discretization of the domain $\Omega \times (0,T)$ (see Definition 2.2). For all $K \in \mathcal{T}$ the initial condition is approximated by

$$u_K^0 = \frac{1}{m(K)} \int_K u_{\mathrm{ini}}(x)\, dx. \tag{2.4}$$

For all $n \in \{0 \dots M\}$, we formally integrate the equations of the system (1.2) over a volume K and over $(n\delta t, (n+1)\delta t)$:

$$\begin{cases} \displaystyle\int_{n\delta t}^{(n+1)\delta t} \int_{\partial K} \mathbf{Q}(x,t).\mathbf{n}(x) d\zeta(x) dt = 0 \\ \displaystyle\int_K \phi(x)\Big(u(x,t_{n+1}) - u(x,t_n)\Big)dx + \\ \displaystyle\int_{n\delta t}^{(n+1)\delta t} \int_{\partial K} \Big(f(u,\mathbf{Q},\mathbf{G})(x,t) - \mathcal{K}(x)\nabla\Big(\varphi(u)\Big)(x,t)\Big).\mathbf{n}(x) d\zeta(x) dt = 0 \end{cases} \tag{2.5}$$

where $\mathbf{n}$ is the unit normal outward to ∂K. For the first equation of (2.5), taking into account the boundary conditions and using a time explicit formulation for the saturations and a time implicit formulation for the pressures, we have

$$\sum_{L \in N(K)} \int_{K|L} \mathbf{Q}(x, t_{n+\frac{1}{2}}).\mathbf{n}_{K,L} d\zeta(x) = 0$$

with $\mathbf{Q}(x, t_{n+\frac{1}{2}}) = \mathcal{K}(x)\Bigg(\Big(\eta_1(u)(x,t_n)\rho_1 + \eta_2(u)(x,t_n)\rho_2\Big)g\nabla z - \eta_T(u)(x,t_n)\nabla\bar{p}(x,t_{n+1})\Bigg)$.

Discretizing the normal gradients with a centered finite difference scheme and writing the approximation of the various terms with respect to their discrete unknowns, we obtain the pressure scheme

$$\sum_{L \in N(K)} Q_{K,L}^{n+1} = 0 \tag{2.6}$$

where

$$Q_{K,L}^{n+1} = \mathcal{K}_{K|L}\left(\left(\eta_{1,K|L}^n\rho_1 + \eta_{2,K|L}^n\rho_2\right)g\delta z_{K,L} - \eta_{T,K|L}^n\delta \overline{p}_{K,L}^{n+1}\right), \tag{2.7}$$

$$\frac{d_{K|L}}{\mathcal{K}_{K|L}} = \frac{1}{\tau_{K|L}}\left(\frac{d_{K,K|L}}{\mathcal{K}(x_K)} + \frac{d_{L,K|L}}{\mathcal{K}(x_L)}\right),$$

$$\frac{d_{K|L}}{\eta_{\alpha,K|L}^n} = \left(\frac{d_{K,K|L}}{\eta_\alpha(u_K^n)} + \frac{d_{L,K|L}}{\eta_\alpha(u_L^n)}\right),$$

$g\delta z_{K,L} = g\nabla z.\overrightarrow{x_K x_L}$, $\delta u_{K,L} = u_L - u_K$.

Now let us define the saturation scheme.
For the second equation of (2.5), we use a time implicit formulation for the saturations, which yields

$$\int_K \phi(x)\Big(u(x,t_{n+1}) - u(x,t_n)\Big)dx + \delta t \sum_{L\in N(K)} \int_{K|L} \Big(\frac{\eta_1}{\eta_T}(u)(x,t_{n+1})\mathbf{Q}(x,t_{n+\frac{1}{2}}) + \frac{\eta_1\eta_2}{\eta_T}(u)(x,t_{n+1})\mathcal{K}(x)(\rho_1-\rho_2)g\nabla z - \mathcal{K}(x)\nabla\varphi(u)(x,t_{n+1})\Big).\mathbf{n}_{K,L}d\zeta(x) = 0.$$

Then the use of centered finite difference schemes for the discretizations of the normal gradients give

$$m(K)\phi_K\frac{u_K^{n+1}-u_K^n}{\delta t} + \sum_{L\in N(K)}\left(\begin{array}{c}\left(\frac{\eta_1}{\eta_T}\right)_{K|L}^{n+1} Q_{K,L}^{n+1} + \left(\frac{\eta_1\eta_2}{\eta_T}\right)_{K|L}^{n+1}\mathcal{K}_{K|L}(\rho_1-\rho_2)g\delta z_{K,L} - \\ \mathcal{K}_{K|L}\Big(\varphi(u_L^{n+1}) - \varphi(u_K^{n+1})\Big)\end{array}\right) = 0$$

where $\phi_K = \dfrac{1}{m(K)}\displaystyle\int_K \phi(x)dx$. We set $G_{K,L} = \mathcal{K}_{K|L}(\rho_1-\rho_2)g\delta z_{K,L}$, $G_{K,\sigma} = m(\sigma)\mathcal{K}(x_K)(\rho_1 - \rho_2)g\nabla z.\mathbf{n}_\sigma$, for $\sigma \subset \partial\Omega$. To compute the upwind terms we consider the following function.

Definition 2.3 *Let $F(a,b,Q,G)$ defined by*

1. if $Q \geq 0$ and $G \leq 0$

$$F(a,b,Q,G) = \begin{cases} \dfrac{\eta_1(a)\Big(Q + G\eta_2(a)\Big)}{\eta_1(a)+\eta_2(a)} & \textit{if } Q + G\eta_2(a) \geq 0, \quad (i) \\ \dfrac{\eta_1(b)\Big(Q + G\eta_2(a)\Big)}{\eta_1(b)+\eta_2(a)} & \textit{otherwise,} \quad (ii) \end{cases}$$

2. if $Q \geq 0$ and $G > 0$

$$F(a,b,Q,G) = \begin{cases} \dfrac{\eta_1(a)\Big(Q + G\eta_2(a)\Big)}{\eta_1(a)+\eta_2(a)} & \textit{if } Q - G\eta_1(a) \geq 0, \quad (i) \\ \dfrac{\eta_1(a)\Big(Q + G\eta_2(b)\Big)}{\eta_1(a)+\eta_2(b)} & \textit{otherwise.} \quad (ii) \end{cases}$$

If $Q < 0$, we set

$$F(a,b,Q,G) = -F(b,a,-Q,-G).$$

Remark 2: Note that F is a nondecreasing Lipschitz continuous function (resp. a nonincreasing Lipschitz continuous function) according to its first argument (resp. according to its second argument). Its Lipschitz constants are bounded by $C_\eta(|Q|+|G|)$ where C_η depends on the mobilities η_α, $\alpha \in \{1,2\}$. For the proof of these results we refer to Enchéry et al. [2002]. □

Computing the transport term thanks to the function F, we obtain the saturation scheme

$$m(K)\phi_K \frac{u_K^{n+1} - u_K^n}{\delta t} + \sum_{L\in N(K)} \left(\begin{array}{c} F(u_K^{n+1}, u_L^{n+1}, Q_{K,L}^{n+1}, G_{K,L}) - \\ \mathcal{K}_{K|L}(\varphi(u_L^{n+1}) - \varphi(u_K^{n+1})) \end{array} \right) = 0. \quad (2.8)$$

In function F mobilities are computed according to an upwind choice. This upwind choice introduces numerical diffusion which can smear out the solution. On the other hand we notice that the capillary pressure also introduces a diffusion which can be used to stabilize the scheme. Thus, trying to center the transport term over the edges where the gradient of φ is sufficient, we improve the precision of the scheme while remaining stable. In practice, the transport term can be computed as a linear combination between the centered and the upwind fluxes. This combination is written thanks to a parameter $0 \le \theta_{K|L}^{n+1} \le 1$ depending on the variable Péclet number on this edge. So we introduce the following function.

Definition 2.4 *Let $\mathcal{F}(\theta,a,b,Q,G)$ defined by*

$$\mathcal{F}(\theta,a,b,Q,G) = \Big(\theta F(a,b,Q,G) + (1-\theta)F(\frac{a+b}{2}, \frac{a+b}{2}, Q, G)\Big) \quad (2.9)$$

where $F(a,b,Q,G)$ is defined in Definition 2.3.

The saturation scheme is thus given by

$$m(K)\phi_K \frac{u_K^{n+1} - u_K^n}{\delta t} + \sum_{L\in N(K)} \left(\begin{array}{c} \mathcal{F}(\theta_{K|L}^{n+1}, u_K^{n+1}, u_L^{n+1}, Q_{K,L}^{n+1}, G_{K,L}) - \\ \mathcal{K}_{K|L}(\varphi(u_L^{n+1}) - \varphi(u_K^{n+1})) \end{array} \right) = 0 \quad (2.10)$$

where

$$\theta_{K|L}^{n+1} = \max\left(0, 1 - \frac{\mathcal{K}_{K|L}(\varphi(u_L^{n+1}) - \varphi(u_K^{n+1}))}{\Lambda_{K,L}^{n+1}(u_K^{n+1}, u_L^{n+1})}\right) \quad (2.11)$$

with $\Lambda_{K,L}^{n+1}(a,b) = F(\frac{a+b}{2}, \frac{a+b}{2}, Q_{K,L}^{n+1}, G_{K,L}) - F(a,b,Q_{K,L}^{n+1}, G_{K,L})$.

2.3 Pressure estimates

In this section, we prove pressure estimates on $(\bar{p}_K^{n+1})_{K\in\mathcal{T},\, n\in\{0\dots M\}}$. We first define the discrete H^1-seminorm.

Definition 2.5 *Let Ω be a domain satisfying* A1-1 *and $\mathcal{M}$ be an admissible mesh in the sense of Definition 2.1. For $u \in \mathcal{X}(\mathcal{M})$, its discrete H^1-seminorm is defined by*

$$|u|_{1,\mathcal{M}} = \Big(\sum_{K|L\in\mathcal{E}_{int}} \tau_{K|L} |\delta u_{K,L}|^2\Big)^{\frac{1}{2}}$$

where $\delta u_{K,L} = u_L - u_K$.

The following proposition states that the discrete H^1-seminorm and the L^2-norm of $(\bar{p}_K^{n+1})_{K\in\mathcal{T},\, n\in\{0\dots M\}}$ remain bounded. These results are obtained by using the same arguments as in Enchéry et al. [2002].

Proposition 2.1 *Under Assumptions 1.1, let $\mathcal{D}$ be an admissible discretization of the domain $\Omega\times(0,T)$ in the sense of Definition 2.2. For all $n\in\{0\dots M\}$, let $\bar{p}_{\mathcal{M}}^{n+1}\in\mathcal{X}(\mathcal{M})$ where $(u_K^{n+1},\bar{p}_K^{n+1})_{K\in\mathcal{T},n\in\{0\dots M\}}$ is a solution to (2.4)-(2.6)- (2.7)-(2.9)-(2.10)-(2.11). Then there exists a constant C_1 depending only on the data and not on $\mathcal{D}$ nor on $(u_K^{n+1},\bar{p}_K^{n+1})_{K\in\mathcal{T},n\in\{0\dots M\}}$, such that*

$$|\bar{p}_{\mathcal{M}}^{n+1}|_{1,\mathcal{M}}\le C_1\,. \tag{2.12}$$

Moreover if we assume, for example, that $\displaystyle\int_\Omega \bar{p}_{\mathcal{M}}^{n+1}(x)dx=0$, there exists C_2 which depends on the same parameters as C_1 such that

$$\left\|\bar{p}_{\mathcal{M}}^{n+1}\right\|_{L^2(\Omega)}\le C_2\,. \tag{2.13}$$

2.4 L^∞ stability

We now prove the L^∞ stability of the saturation calculation.

Proposition 2.2 *Under Assumptions 1.1, let $\mathcal{D}$ be an admissible discretization of $\Omega\times(0,T)$ in the sense of Definition 2.2 and $(u_K^{n+1},p_K^{n+1})_{K\in\mathcal{T},n\in\{0\dots M\}}$ be a solution to (2.4)-(2.6)-(2.7)- (2.9)-(2.10) where, for all $K\in\mathcal{T}$ and $L\in N(K)$, the parameter $\theta_{K|L}^{n+1}$ satisfies*

$$1-\frac{\mathcal{K}_{K|L}(\varphi(u_L^{n+1})-\varphi(u_K^{n+1}))}{\Lambda_{K,L}^{n+1}(u_K^{n+1},u_L^{n+1})}\le\theta_{K|L}^{n+1}\le 1. \tag{2.14}$$

((2.11) and the upwind weighting scheme satisfy condition (2.14).)
Then we have

$$\forall n\in\{0\dots M\},\forall K\in\mathcal{T},\ 0\le u_K^n\le 1. \tag{2.15}$$

Proof:
For all $K\in\mathcal{T}$, we rewrite (2.10) as

$$u_K^n=u_K^{n+1}+\frac{\delta t}{\phi_K m(K)}\sum_{L\in N(K)} f_{K|L}^{n+1}(u_L^{n+1}-u_K^{n+1})+F(u_K^{n+1},u_L^{n+1},Q_{K,L}^{n+1},G_{K,L})$$

with

$$f_{K|L}^{n+1}=\frac{1}{u_L^{n+1}-u_K^{n+1}}\left(\begin{array}{l}(1-\theta_{K|L}^{n+1})\Big(F(\frac{u_K^{n+1}+u_L^{n+1}}{2},\frac{u_K^{n+1}+u_L^{n+1}}{2},Q_{K,L}^{n+1},G_{K,L})-\\ F(u_K^{n+1},u_L^{n+1},Q_{K,L}^{n+1},G_{K,L})\Big)-\mathcal{K}_{K|L}\Big(\varphi(u_L^{n+1})-\varphi(u_K^{n+1})\Big)\end{array}\right).$$

Let us prove (2.15) by induction on n. For $n=0$, according to A1-5 and to the definition of u_K^0 given by (2.4), (2.15) is satisfied. Let us assume that (2.15) is satisfied up to n. If there is some $K\in\mathcal{T}$ such that $u_K^{n+1}<0$ then we have $u_{K_{\min}}^{n+1}=\min_{K\in\mathcal{T}}(u_K^{n+1})<0$. So

$$u^n_{K_{\min}} < u^{n+1}_{K_{\min}} + \frac{\delta t}{\phi_{K_{\min}} m(K_{\min})} \sum_{L\in N(K_{\min})} f^{n+1}_{K_{\min}|L}(u^{n+1}_L - u^{n+1}_{K_{\min}})+$$
$$F(u^{n+1}_{K_{\min}}, u^{n+1}_L, Q^{n+1}_{K_{\min},L}, G_{K_{\min},L}).$$

Moreover the function $F(a, ., Q^{n+1}_{K_{\min},L}, G_{K_{\min},L})$ is nonincreasing and $f^{n+1}_{K_{\min}|L} \leq 0$, so

$$u^n_{K_{\min}} < \frac{\delta t}{\phi_{K_{\min}} m(K_{\min})} \sum_{L\in N(K_{\min})} F(u^{n+1}_{K_{\min}}, u^{n+1}_{K_{\min}}, Q^{n+1}_{K_{\min},L}, G_{K_{\min},L}). \tag{2.16}$$

According to A1-3 we have $\eta_1(u) = 0$ and $\eta_2(u) = \eta_2(0)$ for all $u \leq 0$. Thus

$$F(u^{n+1}_{K_{\min}}, u^{n+1}_{K_{\min}}, Q^{n+1}_{K_{\min},L}, G_{K_{\min},L}) = 0.$$

Consequently (2.16) leads to $u^n_{K_{\min}} < 0$, which is in contradiction with the induction hypothesis. In the same way we prove that for all $K \in \mathcal{T}$, $u^{n+1}_K < 1$. ■

2.5 Existence of a discrete solution

We prove here that the coupled system in pressure and saturation admits at least one solution. This result is established by using a topological degree (see Deimling [1980], Kavian [1993]).

Proposition 2.3 *Under Assumptions 1.1, let $\mathcal{D}$ be an admissible discretization of the domain $\Omega \times (0,T)$ in the sense of Definition 2.2. Then, for all $n \in \{0 \dots M\}$, there exists at least one solution $(u^{n+1}_K, \bar{p}^{n+1}_K)_{K\in\mathcal{T}}$ to (2.4)-(2.6)-(2.7)-(2.9)-(2.10)-(2.11).*

Proof:

Let $n \in \{0 \dots M\}$. The existence and uniqueness (up to an additive constant) of $(\bar{p}^{n+1}_K)_{K\in\mathcal{T}}$ can be proved thanks to the pressure estimate (2.13) and by following the method proposed in Enchéry et al. [2002]. Consequently it suffices to establish the existence of a solution $(u^{n+1}_K)_{K\in\mathcal{T}}$ to (2.4)-(2.9)-(2.10)-(2.11),
$(Q^{n+1}_{K|L})_{K|L\in\mathcal{E}_{int}, n\in\{0\dots M\}}$ be given. This can be done by using a now classical argument based on the topological degree (see Enchéry et al. [2002], Eymard et al. [2000]). The uniqueness of the discrete solution in saturation is obtained thanks to the monotonous properties of the scheme in $L^1(\Omega)$ (see the proof of Proposition 2.2). ■

3 Convergence of the scheme in a simplified case

In this section we prove the convergence of the saturation scheme (2.4)-(2.9)-(2.10) where the total throughput $\mathbf{Q}$ is a given $C^1(\Omega \times (0,T))$-function. This result also holds for the explicit scheme. Throughout this proof we assume that the following hypotheses are fulfilled.

Assumptions 3.1

A3-1. *The total throughput $\mathbf{Q}$ is a given $C^1(\Omega \times (0,T))$-function and satisfies*

$$\forall (x,t) \in \Omega \times (0,T),\ \mathrm{div}(\mathbf{Q})(x,t) = 0,$$
$$\forall (x,t) \in \partial\Omega \times (0,T),\ \mathbf{Q}(x,t).\mathbf{n}(x) = 0.$$

A3-2. *The porosity ϕ and the absolute permeability $\mathcal{K}$ are constant and equal to* 1.

We set $Q_{max} = \|Q\|_{L^\infty(\Omega\times(0,T))}$ and, for all $n \in \{0\dots M\}$ and for all $K|L \in \mathcal{E}_{int}$, $Q_{K,L}^{n+1} = \dfrac{1}{\delta t}\displaystyle\int_{n\delta t}^{(n+1)\delta t}\int_{K|L} \mathbf{Q}(x,t).\mathbf{n}_{K,L}\, d\zeta(x)dt$. Let $0 < \varepsilon < \dfrac{1}{2}$. In this section, we assume that for all $n \in \{0\dots M\}$ and for all $K|L \in \mathcal{E}_{int}$, $\theta_{K|L}^{n+1}$ is given by

$$\theta_{K|L}^{n+1} = \max\left(0, 1 - \frac{(1-2\varepsilon)\tau_{K|L}(\varphi(u_L^{n+1} - \varphi(u_K^{n+1}))}{\Lambda_{K,L}^{n+1}(u_K^{n+1}, u_L^{n+1})}\right). \tag{3.17}$$

The proof lies on the use of the Kolmogorov theorem. Thanks to this theorem we extract a subsequence of solutions in saturation which strongly converges in $L^2(\Omega\times(0,T))$. As the diffusion term is nonlinear, we can not obtain compactness directly from the sequence $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$ but from the sequence $(\varphi(u_{\mathcal{D}_m}))_{m\in\mathbb{N}}$. As the function φ is continuous and strictly increasing, the convergence of a subsequence of $(\varphi(u_{\mathcal{D}_m}))_{m\in\mathbb{N}}$ implies the convergence of a subsequence of $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$.
The application of the Kolmogorov theorem require some estimates. First the L^∞-stability of the saturation calculation ensures that $(\varphi(u_{\mathcal{D}_m}))$ is bounded in $L^2(\Omega\times(0,T))$. But we must also prove that the time and the space translates of this function uniformly vanish as the translation step tends toward 0. The last step of the proof consists in proving that the limit we obtain is solution to a weak problem. So we have the following theorem.

Theorem 3.1 *Under Assumptions 1.1 and 3.1, let us consider a sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of admissible discretizations in the sense of Definition 2.2. We assume that there exists $\alpha > 0$ such that for all $m \in \mathbb{N}$ $\mathrm{regul}(\mathcal{T}_m) \le \alpha$ and such that $\mathrm{size}(\mathcal{D}_m) \to 0$ as $m \to +\infty$. Let $u_{\mathcal{D}_m} = u_m \in \mathcal{X}(\mathcal{D}_m)$ be a solution of the equations (2.4)-(2.9)-(2.10)-(3.17) for $\mathcal{D} = \mathcal{D}_m$. Then there exists a subsequence of approximated solutions which we still denote by $(u_m)_{m\in\mathbb{N}}$ such that*

- *$(u_m)_{m\in\mathbb{N}}$ converges in $L^q(\Omega\times(0,T))$ for all $1 \le q < \infty$ towards a function $u \in L^\infty(\Omega\times(0,T))$ and such that $\varphi(u) \in L^2(0,T,H^1(\Omega))$.*
- *u is a solution to the weak problem: $\forall\psi \in C_{test}$,*

$$\int_0^T\int_\Omega (u\psi_t + f(u,\mathbf{Q},\mathbf{G}).\nabla\psi - \nabla\varphi(u).\nabla\psi)\ dxdt + \int_\Omega u_{ini}\psi(.,0)dx = 0 \tag{3.18}$$

where $C_{test} = \{\psi \in H^1(\Omega\times(0,T))\ /\ \psi(.,T) = 0\}$.

3.1 Space translates

To obtain an upper bound on the time and space translates of the function $\varphi(u_{\mathcal{D}})$, we must first show that the discrete $L^2(0,T,H^1(\Omega))$ semi-norm is bounded and that this bound does not depend on the discretization. We give below the definition of this norm.

3.1.1 Discrete $L^2(0,T,H^1(\Omega))$-seminorm for the function $\varphi(u_{\mathcal{D}})$

Definition 3.1 *Let $\Omega\times(0,T)$ be a domain satisfying* A1-1 *and $\mathcal{D}$ be an admissible discretization of this domain in the sense of Definition 2.2. The $L^2(0,T,H^1(\Omega))$-seminorm of a function $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ is defined by*

$$|u_{\mathcal{D}}|_{1,\mathcal{D}}^2 = \sum_{n=0}^{M}\delta t\sum_{K|L\in\mathcal{E}_{int}}\tau_{K|L}(u_L^n - u_K^n)^2.$$

For the function $\varphi(u_{\mathcal{D}})$ we have the following estimate.

Proposition 3.1 *Under Assumptions 1.1 and 3.1, let $\mathcal{D}$ be an admissible discretization of the domain $\Omega \times (0,T)$ in the sense of Definition 2.2. Let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be given by (2.4)-(2.9)- (2.10)-(3.17). The discrete $L^2(0,T,H^1(\Omega))$-seminorm of $\varphi(u_{\mathcal{D}})$ is bounded by a constant C_3 depending only on u_{ini}, C_η, L_φ, ρ_1, ρ_2, g, ε, Ω, T such that*

$$|\varphi(u_{\mathcal{D}})|^2_{1,\mathcal{D}} \le C_3\,. \tag{3.19}$$

Proof:
Multiplying the equation (2.10) by $\delta t u_K^{n+1}$ and summing over all control volumes $K \in \mathcal{T}$ and over all $n \in \{0 \dots M\}$, we end up with $E_1 + E_2 + E_3 = 0$ where

$$\begin{aligned}
E_1 &= \sum_{n=0}^{M} \sum_{K\in\mathcal{T}} m(K)(u_K^{n+1} - u_K^n)u_K^{n+1} \\
&= \frac{1}{2}\sum_{K\in\mathcal{T}} m(K)(u_K^{M+1})^2 - \frac{1}{2}\sum_{K\in\mathcal{T}} m(K)(u_K^0)^2 + \frac{1}{2}\sum_{n=0}^{M}\sum_{K\in\mathcal{T}} m(K)(u_K^{n+1}-u_K^n)^2 \\
&\ge \frac{1}{2}\sum_{K\in\mathcal{T}} m(K)(u_K^{M+1})^2 - \frac{1}{2}\|u_{ini}\|^2_{L^2(\Omega)}\,, \\
E_2 &= \sum_{n=0}^{M} \delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \begin{pmatrix} \theta_{K|L}^{n+1} F(u_K^{n+1}, u_L^{n+1}, Q_{K,L}^{n+1}, G_{K,L}) + \\ (1-\theta_{K|L}^{n+1}) F(\frac{u_K^{n+1}+u_L^{n+1}}{2}, \frac{u_K^{n+1}+u_L^{n+1}}{2}, Q_{K,L}^{n+1}, G_{K,L}) \end{pmatrix} u_K^{n+1}, \\
E_3 &= \sum_{n=0}^{M} \delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \tau_{K|L}(\varphi(u_K^{n+1}) - \varphi(u_L^{n+1}))u_K^{n+1} \\
&= \sum_{n=0}^{M} \delta t \sum_{K|L\in\mathcal{E}_{int}} \tau_{K|L}(\varphi(u_K^{n+1}) - \varphi(u_L^{n+1}))(u_K^{n+1} - u_L^{n+1}).
\end{aligned}$$

Lower bound on E_2:

First we notice that the assumption A3-1 and the relation $\operatorname{div}(\overrightarrow{G}) = 0$ imply that

$$\forall u \in [0,1],\ \forall K \in \mathcal{T},\ \sum_{L\in N(K)} F(u,u,Q_{K,L}^{n+1}, G_{K,L}) + \sum_{\sigma\in\mathcal{E}_{ext}\bigcap\mathcal{E}_K} F(u,u,0,G_{K,\sigma}) = 0 \tag{3.20}$$

where $G_{K,\sigma} = \tau_\sigma(\rho_1 - \rho_2)g(z_\sigma - z_K)$, z_σ is the depth of the intersection of the line passing through x_K and orthogonal to σ. For $u = u_K^{n+1}$ we multiply (3.20) by u. Substracting this relation to E_2 and gathering with respect to the inner sides we get

$$\begin{aligned}
E_2 = \sum_{n=0}^{M} \delta t \Bigg[&\sum_{K|L\in\mathcal{E}_{int}} \Bigg(\theta_{K|L}^{n+1}(u_L^{n+1} - u_K^{n+1})\Lambda_{K,L}^{n+1}(u_K^{n+1}, u_L^{n+1}) + \\
&\Big(F(\frac{u_K^{n+1}+u_L^{n+1}}{2}, \frac{u_K^{n+1}+u_L^{n+1}}{2}, Q_{K,L}^{n+1}, G_{K,L}) - F(u_K^{n+1}, u_K^{n+1}, Q_{K,L}^{n+1}, G_{K,L})\Big)u_K^{n+1} - \\
&\Big(F(\frac{u_K^{n+1}+u_L^{n+1}}{2}, \frac{u_K^{n+1}+u_L^{n+1}}{2}, Q_{K,L}^{n+1}, G_{K,L}) - F(u_L^{n+1}, u_L^{n+1}, Q_{K,L}^{n+1}, G_{K,L})\Big)u_L^{n+1} \Bigg) - \\
&\sum_{K\in\mathcal{T}} \Bigg(\sum_{\sigma\in\mathcal{E}_{ext}\bigcap\mathcal{E}_K} F(u_K^{n+1}, u_K^{n+1}, 0, G_{K,\sigma})u_K^{n+1} \Bigg) \Bigg].
\end{aligned}$$

Now let us consider $\Phi^n_{K,L}(.)$ a primitive of the function $(.)F'(.,.,Q^{n+1}_{K,L},G_{K,L})$. Integrating by part, we have

$$E_2 = \sum_{n=0}^{M} \delta t \Bigg[\sum_{K|L\in\mathcal{E}_{int}} \Bigg(\theta^{n+1}_{K|L}(u^{n+1}_L - u^{n+1}_K)\Lambda^{n+1}_{K|L}(u^{n+1}_K, u^{n+1}_L) + \Phi^{n+1}_{K,L}(u^{n+1}_L) - \Phi^{n+1}_{K,L}(u^{n+1}_K) + \int_{u^{n+1}_K}^{u^{n+1}_L} \Big(F(s,s,Q^{n+1}_{K,L},G_{K,L}) - F(\frac{u^{n+1}_K + u^{n+1}_L}{2}, \frac{u^{n+1}_K + u^{n+1}_L}{2}, Q^{n+1}_{K,L}, G_{K,L}) \Big) ds \Bigg) - \sum_{K\in\mathcal{T}} \Bigg(\sum_{\sigma\in\mathcal{E}_{ext}\bigcap\mathcal{E}_K} F(u^{n+1}_K, u^{n+1}_K, 0, G_{K,\sigma}) u^{n+1}_K \Bigg) \Bigg].$$

Then we use the following lemma to get a lower bound on the right hand side. Its proof is given in Eymard et al. [2000] pp. 105.

Lemma 3.1 *Let $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ a Lipschitz continuous function which is nondecreasing with respect to its first argument and nonincreasing with respect to its second argument. We denote respectively by G_1 and G_2 its Lipschitz constants with respect to its first and second arguments. Then for all a, b belonging to $\mathbb{R}$ we have*

$$\int_a^b \Big(g(s,s) - g(a,b) \Big) ds \geq \frac{1}{2(G_1+G_2)} \Big((g(b,b) - g(a,b))^2 + (g(a,a) - g(a,b))^2 \Big).$$

Substituting the function $g(.,.)$ by $F(.,.,Q^{n+1}_{K,L},G_{K,L})$, we obtain

$$E_2 \geq \sum_{n=0}^{M} \delta t \Bigg[\sum_{K|L\in\mathcal{E}_{int}} \Bigg((\theta^{n+1}_{K|L} - 1)(u^{n+1}_L - u^{n+1}_K)\Lambda^{n+1}_{K|L}(u^{n+1}_K, u^{n+1}_L) + \Big(\Phi^{n+1}_{K,L}(u^{n+1}_L) - \Phi^{n+1}_{K,L}(u^{n+1}_K) \Big) + \frac{1}{4C_\eta(|Q^{n+1}_{K,L}| + |G_{K,L}|)} \times \Bigg(\Big(F(u^{n+1}_K, u^{n+1}_K, Q^{n+1}_{K,L}, G_{K,L}) - F(u^{n+1}_K, u^{n+1}_L, Q^{n+1}_{K,L}, G_{K,L}) \Big)^2 + \Big(F(u^{n+1}_L, u^{n+1}_L, Q^{n+1}_{K,L}, G_{K,L}) - F(u^{n+1}_K, u^{n+1}_L, Q^{n+1}_{K,L}, G_{K,L}) \Big)^2 \Bigg) \Bigg) - \sum_{K\in\mathcal{T}} \Bigg(\sum_{\sigma\in\mathcal{E}_{ext}\bigcap\mathcal{E}_K} F(u^{n+1}_K, u^{n+1}_K, 0, G_{K,\sigma}) u^n_K \Bigg) \Bigg].$$

But, using again (3.20), we notice that, for all $n \in \{0\dots M\}$ and for all $K \in \mathcal{T}$,

$$\sum_{L\in N(K)} \Phi^n_{K,L}(u^{n+1}_K) = \sum_{L\in N(K)} \Bigg(u^{n+1}_K F(u^{n+1}_K, u^{n+1}_K, Q^{n+1}_{K,L}, G_{K,L}) - \int_0^{u^{n+1}_K} F(s,s,Q^{n+1}_{K,L},G_{K,L})\ ds \Bigg) = - \sum_{\sigma\in\mathcal{E}_{ext}\bigcap\mathcal{E}_K} \Bigg(u^{n+1}_K F(u^{n+1}_K, u^{n+1}_K, 0, G_{K,\sigma}) - \int_0^{u^{n+1}_K} F(s,s,0,G_{K,\sigma})\ ds \Bigg).$$

Therefore

$$E_2 \geq \sum_{n=0}^{M} \delta t \Bigg[\sum_{K|L \in \mathcal{E}_{int}} \Bigg((\theta^{n+1}_{K|L} - 1)(u^{n+1}_L - u^{n+1}_K)\Lambda^{n+1}_{K|L}(u^{n+1}_K, u^{n+1}_L) +$$

$$\frac{1}{4C_\eta(|Q^{n+1}_{K,L}| + |G_{K,L}|)} \Bigg(\Big(F(u^{n+1}_K, u^{n+1}_K, Q^{n+1}_{K,L}, G_{K,L}) - F(u^{n+1}_K, u^{n+1}_L, Q^{n+1}_{K,L}, G_{K,L}) \Big)^2 +$$

$$\Big(F(u^{n+1}_L, u^{n+1}_L, Q^{n+1}_{K,L}, G_{K,L}) - F(u^{n+1}_K, u^{n+1}_L, Q^{n+1}_{K,L}, G_{K,L}) \Big)^2 \Bigg) \Bigg) -$$

$$\sum_{K \in \mathcal{T}} \Bigg(\sum_{\sigma \in \mathcal{E}_{ext} \bigcap \mathcal{E}_K} \int_0^{u^{n+1}_K} F(s, s, 0, G_{K,\sigma}) \Bigg) \Bigg].$$

Moreover we notice that

$$\operatorname{sign}\Big(\Lambda^{n+1}_{K,L}(u^{n+1}_K, u^{n+1}_L)\Big) = \operatorname{sign}\Big(\varphi(u^{n+1}_K) - \varphi(u^{n+1}_L)\Big) \tag{3.21}$$

and that (3.17) yields

$$(1 - \theta^{n+1}_{K|L})\Big|\Lambda^{n+1}_{K,L}(u^{n+1}_K, u^{n+1}_L)\Big| \leq (1 - \varepsilon)\tau_{K|L}\Big|\varphi(u^{n+1}_K) - \varphi(u^{n+1}_L)\Big|. \tag{3.22}$$

So, collecting the previous lower and upper bounds, using (3.21), (3.22), and noticeing that φ is Lipschitz continuous we end up with

$$\varepsilon \sum_{n=0}^{M} \delta t \Bigg[\sum_{K|L \in \mathcal{E}_{int}} \Bigg(\frac{1}{4C_\eta(|Q^{n+1}_{K,L}| + |G_{K,L}|)} \times$$

$$\Bigg(\Big(F(u^{n+1}_K, u^{n+1}_K, Q^{n+1}_{K,L}, G_{K,L}) - F(u^{n+1}_K, u^{n+1}_L, Q^{n+1}_{K,L}, G_{K,L}) \Big)^2 +$$

$$\Big(F(u^{n+1}_L, u^{n+1}_L, Q^{n+1}_{K,L}, G_{K,L}) - F(u^{n+1}_K, u^{n+1}_L, Q^{n+1}_{K,L}, G_{K,L}) \Big)^2 \Bigg) +$$

$$\frac{\tau_{K|L}}{L_\varphi} \Big(\varphi(u^{n+1}_L) - \varphi(u^{n+1}_K) \Big)^2 \Bigg) - \sum_{K \in \mathcal{T}} \Bigg(\sum_{\sigma \in \mathcal{E}_{ext} \bigcap \mathcal{E}_K} \int_0^{u^{n+1}_K} F(s, s, 0, G_{K,\sigma}) ds \Bigg) \Bigg]$$

$$\leq \frac{1}{2} ||u_{\text{ini}}||^2_{L^2(\Omega)}.$$

Thus

$$\frac{\varepsilon}{L_\varphi} |\varphi(u_\mathcal{D})|^2_{1,\mathcal{D}} - \Bigg| \sum_{n=0}^{M} \delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{ext} \bigcap \mathcal{E}_K} \int_0^{u^{n+1}_K} F(s, s, 0, G_{K,\sigma}) ds \Bigg| \leq \frac{1}{2} ||u_{\text{ini}}||^2_{L^2(\Omega)}.$$

The term with the integral in the left hand side can be bounded in the following way

$$\Bigg| \sum_{n=0}^{M} \delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{ext} \bigcap \mathcal{E}_K} \int_0^{u^{n+1}_K} F(s, s, 0, G_{K,\sigma}) ds \Bigg| \leq 2TC_\eta(\rho_2 - \rho_1) g m(\partial\Omega).$$

So it leads to

$$\frac{\varepsilon}{L_\varphi} |\varphi(u_\mathcal{D})|^2_{1,\mathcal{D}} \leq \frac{1}{2} ||u_{\text{ini}}||^2_{L^2(\Omega)} + 2TC_\eta(\rho_2 - \rho_1) g m(\partial\Omega),$$

which gives the result taking $C_3 = \frac{L_\varphi}{\varepsilon} \Bigg(\frac{1}{2} \|u_{\text{ini}}\|^2_{L^2(\Omega)} + 2TC_\eta(\rho_2 - \rho_1) g m(\partial\Omega) \Bigg)$. ■

3.1.2 Upper bound of the space translates of the function $\varphi(u_{\mathcal{D}})$

We conclude this section dedicated to the space translates with the following proposition proved in Eymard et al. [2000] pp. 74–75.

Proposition 3.2 *Under Assumptions 1.1 and 3.1, let $\mathcal{D}$ be an admissible discretization of the domain $\Omega \times (0,T)$ in the sense of Definition 2.2. Let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be given by (2.4)-(2.9)-(2.10)-(3.17). Let $\xi \in \mathbb{R}^d$ and Ω_ξ be a subset of Ω defined by $\Omega_\xi = \{x \in \Omega \ / \ [x, x+\xi] \subset \Omega\}$. Then there exists a constant C_4 depending only on the number of edges of Ω such that the function $\varphi(u_{\mathcal{D}})$ satisfies*

$$\int_0^T \int_{\Omega_\xi} |\varphi(u_{\mathcal{D}}(x+\xi,t) - \varphi(u_{\mathcal{D}}(x,t)|^2 dxdt \leq |\xi| \Big(|\xi| + C_4 \, size(\mathcal{T}) \Big) |\varphi(u)|^2_{1,\mathcal{D}}. \tag{3.23}$$

Moreover if we set $u_{\mathcal{D}} = 0$ for all $(x,t) \notin \Omega \times (0,T)$ then, for all $\xi \in \mathbb{R}^d$, we have

$$||\varphi(u_{\mathcal{D}}(.+\xi,.)) - \varphi(u_{\mathcal{D}}(.,.))||^2_{L^2(\mathbb{R}^{d+1})} \leq C_5 \, |\xi| \Big(|\xi| + size(\mathcal{M}) + 1 \Big) \tag{3.24}$$

where C_5 depends on the constants C_3, C_4, T, L_φ and on the domain Ω.

3.2 Time translates

We now prove that the time translates of the function $\varphi(u_{\mathcal{D}})$ remain bounded.

Proposition 3.3 *Under Assumptions 1.1 and 3.1 let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.2. Let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be the solution of equations (2.4)-(2.9)-(2.10)-(3.17). Outside the domain $\Omega \times (0,T)$, we set $u_{\mathcal{D}} = 0$. Then, for all $\tau \in \mathbb{R}$, the following inequality holds*

$$||\varphi(u_{\mathcal{D}}(x,t+\tau) - \varphi(u_{\mathcal{D}}(x,t)||^2_{L^2(\mathbb{R}^{d+1})} \leq C_6 \, |\tau| \tag{3.25}$$

where C_6 depends on L_φ, C_η, C_3, d, Ω, T, $Q_{\max}$, ρ_α, $\alpha \in \{1,2\}$ and g.

Proof:
We obtain the result by using the estimate (3.19) and by following the method presented in Eymard et al. [2000] pp. 106–108. ■

3.3 Proof of the convergence Theorem 3.1

Convergence of a subsequence of $(u_m)_{m\in\mathbb{N}}$:

We set $\tilde{u}_m = u_m$ on $\Omega \times (0,T)$ and we extend this function to 0 on $\mathbb{R}^{d+1} \setminus (\Omega \times (0,T))$. Since the function φ is continuous and since $\tilde{u}_m \in [0,1]$ on $\mathbb{R}^{d+1}$ for all $m \in \{0 \dots M\}$, the sequence $\varphi(\tilde{u}_m)$ is bounded in $L^q(\mathbb{R}^{d+1})$ for all $1 \leq q \leq +\infty$. From inequalities (3.24) and (3.25), we deduce that, for all $\xi \in \mathbb{R}^d$ and $\tau \in \mathbb{R}$, there exists a constant $C(\xi,\tau) \to 0$ as $\xi \to 0$ and $\tau \to 0$ such that

$$\|\varphi(\tilde{u}_m(x+\xi,t+\tau)) - \varphi(\tilde{u}_m(x,t))\|^2_{L^q(\mathbb{R}^{d+1})} \leq C(\xi,\tau)$$

for all $1 \leq q < \infty$. Under these conditions, we can apply the Kolmogorov theorem (see Eymard et al. [2000]) and deduce that the sequence $(\varphi(u_m))_{m\in\mathbb{N}}$ is relatively compact in $L^q(\Omega \times (0,T))$. So there exists a subsequence, still denoted by $(\varphi(u_m))_{m\in\mathbb{N}}$ which converges in $L^q(\Omega \times (0,T))$. As φ is a strictly increasing C^1-function, we also deduce the convergence of the sequence (u_m) toward a function $u \in L^q(\Omega \times (0,T)) \cap L^\infty(\Omega \times (0,T))$.

$\varphi(u) \in L^2(0,T,H^1(\Omega))$: see Eymard et al. [2000] pp. 91.

Convergence of u_m toward a weak solution of (3.18):

Let us consider the set $\tilde{C}_{test} = \{\psi \in C^2(\overline{\Omega} \times [0,T]) \ / \ \psi(.,T) = 0\}$ which is dense in C_{test}. Let $\psi \in \tilde{C}_{test}$ and $(u_m)_{m\in\mathbb{N}}$ be the sequence of solutions to (2.4)-(2.9)-(2.10)- (3.17). For all $n \in \{0 \ldots M\}$ and $K \in \mathcal{T}$, we multiply (2.10) by $\psi_K^n = \psi(x_K, n\delta t)$ and we sum these equalities over all the volumes:

$$\begin{aligned}
&\sum_{n=0}^{M}\sum_{K\in\mathcal{T}} m(K)(u_K^{n+1}-u_K^n)\psi_K^n + \\
&\delta t \sum_{L\in N(K)} \Big(\mathcal{F}(\theta_{K|L}^{n+1}, u_K^{n+1}, u_L^{n+1}, Q_{K,L}^{n+1}, G_{K,L}) - \tau_{K|L}(\varphi(u_L^{n+1}) - \varphi(u_K^{n+1}))\Big)\psi_K^n = \\
&E_{1,m} + E_{2,m} + E_{3,m} = 0
\end{aligned}$$

where

$$\begin{aligned}
E_{1,m} &= \sum_{n=0}^{M}\sum_{K\in\mathcal{T}} m(K)(u_K^{n+1}-u_K^n)\psi_K^n, \\
E_{2,m} &= \sum_{n=0}^{M}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \Big(\mathcal{F}(\theta_{K|L}^{n+1}, u_K^{n+1}, u_L^{n+1}, Q_{K,L}^{n+1}, G_{K,L})\Big)\psi_K^n, \\
E_{3,m} &= -\sum_{n=0}^{M}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \tau_{K|L}\Big(\varphi(u_L^{n+1}) - \varphi(u_K^{n+1})\Big)\psi_K^n.
\end{aligned}$$

Convergence of $E_{1,m}$: Following Eymard et al. [2000] pp. 110, we get

$$\lim_{m\to+\infty} E_{1,m} = \int_0^T \int_\Omega u(x,t)\psi_t(x,t)\,dxdt - \int_\Omega u_{\text{ini}}(x)\psi(x,0)\,dx.$$

Convergence of $E_{2,m}$:

Let $F_{2,m} = \int_0^T \int_\Omega f(u_m, \mathbf{Q}, \mathbf{G}).\nabla\psi dxdt$.
According to A3-1 we have

$$\begin{cases} \text{div}(\mathbf{Q}) = 0 \text{ on } \Omega \times (0,T), \\ \mathbf{Q}.\mathbf{n} = 0 \text{ on } \partial\Omega \times (0,T), \\ \text{div}(\mathbf{G}) = 0. \end{cases} \tag{3.26}$$

Using these properties, the term $F_{2,m}$ can be rewritten under the form

$$\begin{aligned}
F_{2,m} = \ & \sum_{n=0}^{M}\delta t\Big(\sum_{K\in\mathcal{T}}\Big(\sum_{L\in N(K)} m(K|L) f(u_K^{n+1}, (\tilde{\psi\mathbf{Q}})_{K,L}^{n+1}, (\tilde{\psi\mathbf{G}})_{K,L}^{n+1}) + \\
& \sum_{\sigma\in\mathcal{E}_K\bigcap\mathcal{E}_{ext}} m(\sigma) f(u_K^{n+1}, 0, (\tilde{\psi\mathbf{G}})_\sigma^{n+1})\Big)\Big)
\end{aligned}$$

where

$$(\tilde{\psi \mathbf{u}})_\sigma^n = \frac{1}{\delta t\, m(\sigma)} \int_{n\delta t}^{(n+1)\delta t} \int_\sigma \psi \mathbf{u}(x,t).\mathbf{n}_\sigma \; d\zeta(x) dt.$$

Now let us consider the terme $\tilde{F}_{i,2,m}$ given by

$$\tilde{F}_{2,m} = \sum_{n=0}^{M} \delta t \Big(\sum_{K\in\mathcal{T}} \Big(\sum_{L\in N(K)} m(K|L) f(u_K^{n+1}, \frac{Q_{K,L}^{n+1}}{m(K|L)}, \frac{G_{K,L}}{m(K|L)}) \psi_{K|L}^{n+1} + \sum_{\sigma\in\mathcal{E}_K \bigcap \mathcal{E}_{ext}} m(\sigma) f(u_K^{n+1}, 0, \frac{G_{K,\sigma}}{m(\sigma)}) \psi_\sigma^{n+1} \Big) \Big)$$

with

$$\psi_\sigma^{n+1} = \frac{1}{\delta t\, m(\sigma)} \int_{n\delta t}^{(n+1)\delta t} \int_\sigma \psi(x,t) d\zeta(x) dt.$$

In a first time, we prove that the difference $|F_{2,m} - \tilde{F}_{2,m}|$ vanishes as $m \to +\infty$.
Gathering the terms according to the inner sides, the difference $dF_{2,m}$ between $\tilde{F}_{2,m}$ and $F_{2,m}$ is such that $dF_{2,m} = dF_{2,\mathrm{m,int}} + dF_{2,\mathrm{m,ext}}$ with

$$\begin{aligned}
dF_{2,\mathrm{m,int}} = & \sum_{n=0}^{M} \delta t \Bigg(\sum_{K|L\in\mathcal{E}_{int}} m(K|L) \times \\
& \Big(f\Big(u_K^{n+1}, \frac{Q_{K,L}^{n+1}}{m(K|L)} \psi_{K|L}^{n+1} - (\tilde{\psi \overrightarrow{Q}})_{K,L}^{n+1}, \frac{G_{K,L}}{m(K|L)} \psi_{K|L}^{n+1} - (\tilde{\psi \overrightarrow{G}})_{K,L}^{n+1} \Big) - \\
& f\Big(u_L^{n+1}, \frac{Q_{K,L}^{n+1}}{m(K|L)} \psi_{K|L}^{n+1} - (\tilde{\psi \overrightarrow{Q}})_{K,L}^{n+1}, \frac{G_{K,L}}{m(K|L)} \psi_{K|L}^{n+1} - (\tilde{\psi \overrightarrow{G}})_{K,L}^{n+1} \Big) \Big) \Bigg), \\
dF_{2,\mathrm{m,ext}} = & \sum_{n=0}^{M} \delta t \sum_{K\in\mathcal{T}} \Bigg(\sum_{\sigma\in\mathcal{E}_K \bigcap \mathcal{E}_{ext}} m(\sigma) \times \\
& \Big(f\Big(u_K^{n+1}, 0, \frac{G_\sigma}{m(\sigma)} \psi_\sigma^{n+1} - (\tilde{\psi \overrightarrow{G}})_\sigma^{n+1} \Big) \Big) \Bigg).
\end{aligned}$$

We then have

$$\begin{aligned}
|dF_{2,\mathrm{m,int}}| \le \sum_{n=0}^{M} \delta t \Bigg(& \sum_{K|L\in\mathcal{E}_{int}} m(K|L) |u_L^{n+1} - u_K^{n+1}| \times \\
& 2C_\eta \Big(\Big| \frac{Q_{K,L}^{n+1}}{m(K|L)} \psi_{K|L}^{n+1} - (\tilde{\psi \overrightarrow{Q}})_{K,L}^{n+1} \Big| + \Big| \frac{G_{K,L}}{m(K|L)} \psi_{K|L}^{n+1} - (\tilde{\psi \overrightarrow{G}})_{K,L}^{n+1} \Big| \Big) \Bigg).
\end{aligned}$$

As functions ψ and $\overrightarrow{Q}$ are smooth, there is a constant C_7 such that

$$|dF_{2,\mathrm{m,int}}| \le C_7 \sum_{n=0}^{M} \delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} m(K|L) \mathrm{diam}(K) |u_L^{n+1} - u_K^{n+1}|.$$

We then use the following Lemma.

Lemma 3.2 *Under Assumptions 1.1 and 3.1, let* $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *be a sequence of admissible discretizations of the domain* $\Omega\times(0,T)$ *in the sense of Definition 2.2 such that* $\mathrm{size}(\mathcal{D}_\mathrm{m})\to 0$ *as* $m\to+\infty$ *and such that there exists* $\alpha>0$ *satisfying* $\mathrm{regul}(\mathcal{M}_\mathrm{m})\leq d\alpha$. *Let* $u_m\in\mathcal{X}(\mathcal{D}_m)$ *be given by equations (2.4)-(2.9)-(2.10)- (3.17). So we have*

$$\sum_{n=0}^{M}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} m(K|L)\mathrm{diam}(K)|u_L^{n+1}-u_K^{n+1}|\to 0 \text{ as } m\ \to+\infty. \tag{3.27}$$

Proof:
This lemma may be easily deduced from the convergence of $u_m\in\mathcal{X}(\mathcal{D}_m)$ toward u in $L^1(\Omega\times(0,T))$ and from the density of the space $C^\infty(\bar{\Omega}\times(0,T))$ in $L^1(\Omega\times(0,T))$. ■

This result ensures that $|dF_{2,\mathrm{m,int}}|\to 0$ as $m\ \to+\infty$. For the term $dF_{2,\mathrm{m,ext}}$ we have thanks to the regularity of ψ and $\mathbf{Q}$ and thanks to the L^∞-stability of the scheme

$$|dF_{2,\mathrm{m,ext}}|\leq C_8\,\mathrm{size}(\mathcal{M}).$$

Consequently $|F_{2,m}-\tilde{F}_{2,m}|\to 0$ as $m\to+\infty$. On the other hand, as $\mathbf{Q}$ and $\mathbf{G}$ are bounded in $L^\infty(\Omega\times(0,T))$, $f(.,\mathbf{Q},\mathbf{G})$ is Lipschitz continuous. Moreover, as $u_m\to u$ in $L^1(\Omega\times(0,T))$ we have

$$F_{2,m}\to\int_0^T\int_\Omega f(u,\mathbf{Q},\mathbf{G}).\nabla\psi dxdt \text{ as } m\to+\infty.$$

Using again the relations (3.26), we rewrite $\tilde{F}_{2,m}$ as

$$\tilde{F}_{2,m}=\sum_{n=0}^{M}\delta t\Big(\sum_{K\in\mathcal{T}}\ \Big(\sum_{L\in N(K)} f(u_K^{n+1},Q_{K,L}^{n+1},G_{K,L})(\psi_{K|L}^{n+1}-\psi_K^n)+ \sum_{\sigma\in\mathcal{E}_K\bigcap\mathcal{E}_{ext}} f(u_K^{n+1},0,G_{K,\sigma})(\psi_\sigma^{n+1}-\psi_K^n)\Big)\Big).$$

But we notice that

$$\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} F(u_K^{n+1},u_L^{n+1},Q_{K,L}^{n+1},G_{K,L})\psi_{K|L}^{n+1}=0.$$

So $E_{2,m}$ can be put under the form

$$E_{2,m}=\sum_{n=0}^{M}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\Big(F(u_K^{n+1},u_L^{n+1},Q_{K,L}^{n+1},G_{K,L})\Big)\Big(\psi_K^n-\psi_{K|L}^{n+1}\Big).$$

We then have

$$|E_{2,m}+\tilde{F}_{2,m}|\leq\sum_{n=0}^{M}\delta t\quad\Big(\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} C_\eta(|Q_{K,L}^{n+1}|+|G_{K,L}|)|u_L^{n+1}-u_K^{n+1}||\psi_K^n-\psi_{K|L}^{n+1}|+ \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K\bigcap\mathcal{E}_{ext}} |f(u_K^n,0,G_{K,\sigma})||\psi_\sigma^{n+1}-\psi_K^n|\Big).$$

So there is a constant C_ψ such that

$$|E_{2,m} + \tilde{F}_{2,m}| \le \sum_{n=0}^{M} \delta t \quad \Big(\sum_{K\in\mathcal{T}} \sum_{L\in N(K)} C_\eta \Big(Q_{max} + (\rho_2 - \rho_1)g\Big) m(K|L)|u_L^{n+1} - u_K^{n+1}| \times$$
$$C_\psi \mathrm{diam(K)} + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K \bigcap \mathcal{E}_{ext}} 2C_\eta(\rho_2 - \rho_1) g m(\sigma) C_\psi \mathrm{diam(K)} \Big).$$

Moreover

$$\sum_{n=0}^{M} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K \bigcap \mathcal{E}_{ext}} C_\eta(\rho_2 - \rho_1) g m(\sigma) C_\psi \mathrm{diam(K)} \le T m(\partial\Omega) C_\eta (\rho_2 - \rho_1) g C_\psi \mathrm{size}(\mathcal{M}).$$

Thus finally we have

$$\lim_{m\to\infty} E_{2,m} = -\int_0^T \int_\Omega f(u, \mathbf{Q}, \mathbf{G}).\nabla\psi dx dt.$$

Convergence of $E_{3,m}$: By applying the method presented in Eymard and Gallouët [2003] we get

$$\lim_{m\to\infty} E_{3,m} = \int_0^T \int_\Omega \nabla\varphi(u)(x,t).\nabla\psi(x,t)\, dx dt.$$

4 Numerical tests

4.1 Numerical data

In this section we detail the numerical data used in the two following tests. The tests have been achieved thanks to a prototype designed for sedimentary basin simulations. Consequently the dimensions of the domain are given in meter and the time is counted in millions of years.

Gravity: $g = 9.81\, m.s^{-2}$

Properties of the fluids:

Type of data	Oil	Water
μ_α	$5.10^{-3}\, Pa.s$	$10^{-3}\, Pa.s$
ρ_α	$800\, kg.m^{-3}$	$1100\, kg.m^{-3}$
$kr_\alpha(u)$	$\begin{cases} 0 & \text{if } u \le 0, \\ u & \text{if } 0 \le u \le 1, \\ 1 & \text{otherwise.} \end{cases}$	$\begin{cases} 1 & \text{if } u \le 0 \\ 1-u & \text{if } 0 \le u \le 1 \\ 0 & \text{otherwise.} \end{cases}$
η_α	$\dfrac{kr_1(u)}{\mu_1}$	$\dfrac{kr_2(u)}{\mu_2}$
$\pi(u)$	$\forall\, 0 \le u \le 1,\ 0.3u$	

Properties of the rock:

- $\phi = 0.1$
- $\mathcal{K} = 50\, \mu D$ $(1\ \mu D = 0.98.10^{-15}\ m^2)$

4.2 Test 1

For a 1-D case, we compare the evolution of the saturation over the time for different schemes. Among these schemes, we have both implicit and explicit variable Péclet number schemes, their upwind equivalents where for all $K|L \in \mathcal{E}_{int}$ $\theta^{n+1}_{K|L} = 1$ and a MUSCL scheme.
Let $\Omega = (0, 3000)$ and $D = (2600, 3000)$. The initial condition u_{ini} is given by

$$u_{\text{ini}}(x) = \begin{cases} 1.0 & \text{if } x \in D \\ 0.0 & \text{otherwise.} \end{cases}$$

Since the boundary is impermeable, the total throughput is equal to 0 on $\Omega \times (0, T)$. The domain is meshed by a cartesian regular grid, $\mathcal{M}$. We denote by $h = \Delta z$ the space step and $N = \text{card}(\mathcal{T})$. The MUSCL scheme we use to discretize (1.2) is defined, in the general case and for all $i \in \{1 \dots N\}$, by

$$\Delta z \phi \frac{u_i^{n+1} - u_i^n}{\delta t} + F^{n+1}_{i+\frac{1}{2}} - F^{n+1}_{i-\frac{1}{2}} + \mathcal{K}\frac{\varphi(u^{n+1}_{i+1}) - \varphi(u_i^{n+1})}{\Delta z} - \mathcal{K}\frac{\varphi(u_i^{n+1}) - \varphi(u^{n+1}_{i-1})}{\Delta z} = 0$$

where

- $$F^{n+1}_{i+\frac{1}{2}} = \begin{cases} \dfrac{\tilde{\eta}^{n+1}_{o,i+\frac{1}{2}}\left(Q^{n+1}_{i+\frac{1}{2}} + G\tilde{\eta}^{n+1}_{w,i+\frac{1}{2}}\right)}{\tilde{\eta}^{n+1}_{o,i+\frac{1}{2}} + \tilde{\eta}^{n+1}_{w,i+\frac{1}{2}}} & \text{for all } i \in \{1 \dots N-1\}, \\ 0 & \text{for } i = 0 \text{ or } i = N, \end{cases}$$

- $G = \mathcal{K}(\rho_1 - \rho_2)g$ and $Q^{n+1}_{i+\frac{1}{2}}$ is given by (2.6)-(2.7),

- **if** $Q^{n+1}_{i+\frac{1}{2}} < 0$

$$\begin{aligned} \tilde{\eta}^{n+1}_{1,i+\frac{1}{2}} &= \tilde{\eta}^{n+1,+}_{1,i+\frac{1}{2}}, \\ \tilde{\eta}^{n+1}_{2,i+\frac{1}{2}} &= \begin{cases} \tilde{\eta}^{n+1,-}_{2,i+\frac{1}{2}} & \text{if } Q^{n+1}_{i+\frac{1}{2}} - G\tilde{\eta}^{n+1}_{o,i+\frac{1}{2}} \geq 0, \\ \tilde{\eta}^{n+1,+}_{2,i+\frac{1}{2}} & \text{otherwise,} \end{cases} \end{aligned}$$

  if $Q^{n+1}_{i+\frac{1}{2}} \geq 0$

$$\begin{aligned} \tilde{\eta}^{n+1}_{2,i+\frac{1}{2}} &= \tilde{\eta}^{n+1,-}_{2,i+\frac{1}{2}}, \\ \tilde{\eta}^{n+1}_{1,i+\frac{1}{2}} &= \begin{cases} \tilde{\eta}^{n+1,-}_{1,i+\frac{1}{2}} & \text{if } Q^{n+1}_{i+\frac{1}{2}} + G\tilde{\eta}^{n+1}_{w,i+\frac{1}{2}} \geq 0, \\ \tilde{\eta}^{n+1,+}_{1,i+\frac{1}{2}} & \text{otherwise,} \end{cases} \end{aligned}$$

- for all $\alpha \in \{1, 2\}$

$$\begin{aligned} \tilde{\eta}^{n+1,+}_{\alpha,i+\frac{1}{2}} &= \eta_\alpha(u^n_{i+1}) - d(z_{i+\frac{1}{2}}, z_{i+1})\delta\eta^n_{\alpha,i+1}, \\ \tilde{\eta}^{n+1,-}_{\alpha,i+\frac{1}{2}} &= \eta_\alpha(u^n_i) + d(z_i, z_{i+\frac{1}{2}})\delta\eta^n_{\alpha,i}, \end{aligned}$$

- $$\delta\eta^n_{\alpha,i} = \begin{cases} \text{sign}(\delta\hat{\eta}^n_{\alpha,i}) \min\left(|\delta\hat{\eta}^n_{\alpha,i}|, \dfrac{|\eta_\alpha(u^n_{i+1}) - \eta_\alpha(u^n_i)|}{d(z_i, z_{i+1})}, \dfrac{|\eta_\alpha(u^n_i) - \eta_\alpha(u^n_{i-1})|}{d(z_{i-1}, z_i)}\right) \\ \text{if these three values have the same sign,} \\ 0 \text{ otherwise,} \end{cases}$$

•

$$\delta\hat{\eta}^n_{\alpha,i} = \frac{|\eta_\alpha(u^n_{i+1}) - \eta_\alpha(u^n_{i-1})|}{d(z_{i-1}, z_{i+1})}.$$

For $h = 50\,m$, Figure 1 shows the rise of oil under gravity along the column Ω at $t = 0.5$. On this figure, we put the saturations computed by the explicit variable Péclet number scheme, the implicit variable Péclet number scheme, the explicit upwind scheme (i.e. $\theta_{K|L} = 1$, $\forall K|L \in \mathcal{E}_{int}$), the implicit upwind scheme and the explicit upwind scheme with slope limiters which we have previously detailed. With this first test, we notice that the precision reached by the variable Péclet number scheme is close to the precision of the MUSCL scheme. The implicit form of this scheme seems to be more diffusive but its solution remains better than the upwind schemes and it needs not so many time steps to compute the solution as all other explicit schemes.

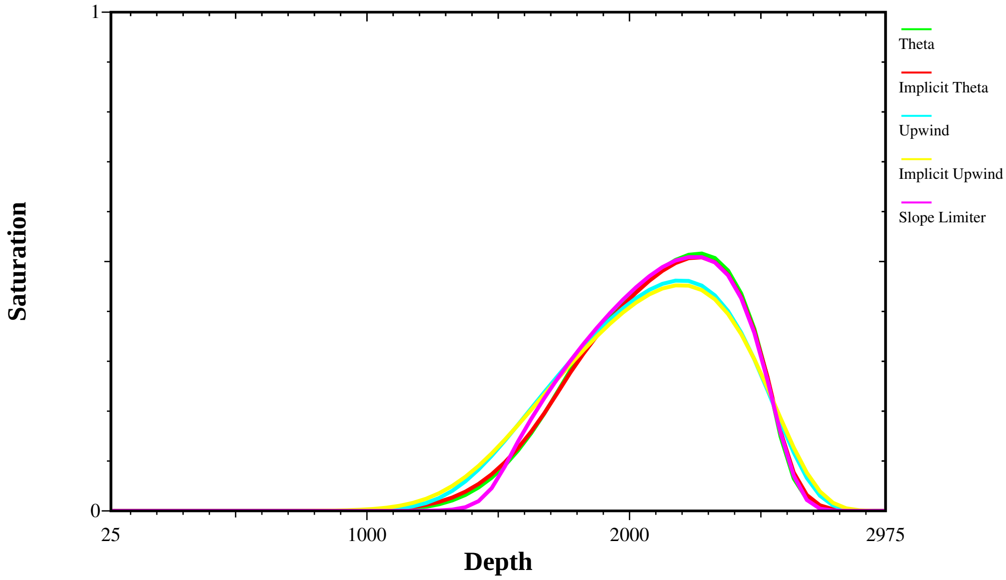

Figure 1: Saturations computed with different schemes at $t = 0.5$.

4.3 Test 2

In this test we determine the numerical convergence rate of the explicit variable Péclet number scheme. We use the same data as the previous test apart from the domain which is smaller and defined by $\Omega = (-500, 0)$ and $D = (-500, -100)$. To evaluate the convergence speed, we have computed a reference solution at $t = 0.1$ for a space step equal to $h = 0.5\,m$. Figure 2 shows the error we obtain for different space steps. The convergence rate is here equal to 1.1342. This result confirms again the better precision of the scheme in comparison with the upwind schemes whose convergence rate does not exceed 1.0.

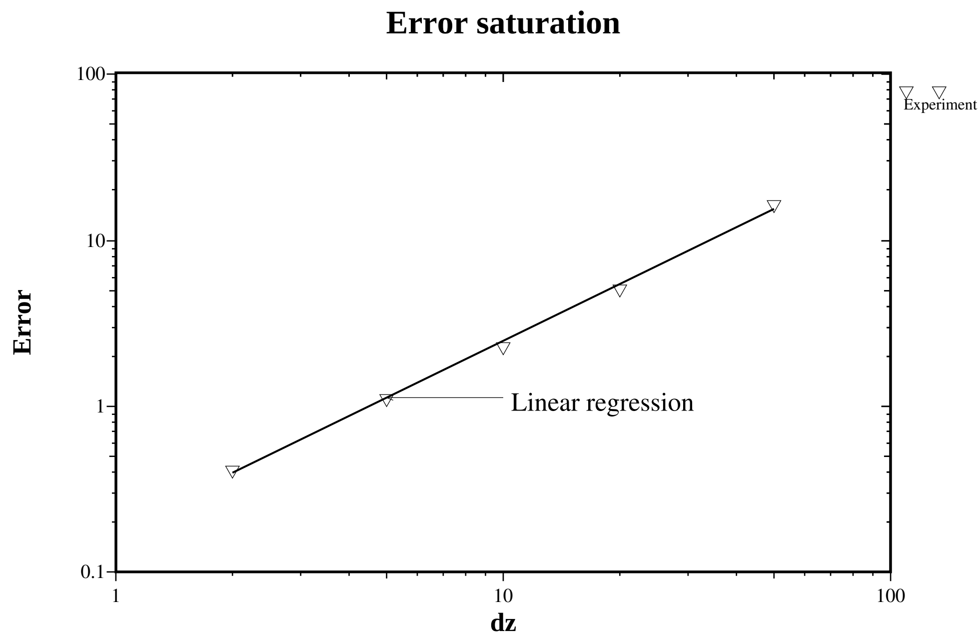

Figure 2: Numerical convergence in saturation

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