NUMERICAL APPROXIMATION OF A TWO-PHASE FLOW PROBLEM IN A POROUS MEDIUM WITH DISCONTINUOUS CAPILLARY FORCES

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Abstract. We consider a simplified model of a two-phase flow through a heterogeneous porous medium. Focusing on the capillary forces motion, a nonlinear degenerate parabolic problem is approximated in a domain shared in two homogeneous parts, each of them being characterized by its relative permeability and capillary curves functions of the phase saturations. We first give a weak form of the conservation equations on the whole domain, with a new general expression of the conditions at the interface between the two regions. We then propose a finite volume scheme for the approximation of the solution, which is shown to converge to a weak solution in 1D, 2D or 3D domains. We conclude with presenting some numerical tests.

Key words. Flows in porous media, Capillarity, Nonlinear PDE of parabolic type, Finite volume methods.

AMS subject classifications. 76S05, 76B45, 35K55, 74S10

1. Introduction. Simulations of two-phase flows through heterogeneous porous media are widely used in petroleum engineering. For example, for exploration purposes, the basin modeling aims to reconstruct the geological history of a sedimentary basin and in particular the migration of hydrocarbon components at geological time scale. The reservoir simulation is devoted to the understanding and the prediction of fluid flows occurring during production processes.
One of the most important consequences of the presence of heterogeneities in a porous medium is the phenomenon of capillary entrapment. This phenomenon occurs at the interface between two geological layers where discontinuous capillary thresholds appear. Indeed if the mean pore radius in one layer is smaller than in the other, the oil phase must reach an access pressure so that the oil phase can enter the least permeable layer. In a sedimentary basin, this mechanism can induce the formation of oilfields. On the other hand, in reservoir engineering, the capillary trapping can reduce the recovery factor since large quantities of oil can remain trapped. Therefore, for this kind of applications, one need a precise understanding of this phenomenon on the physical plane as on the mathematical plane as well.
The physical principles which govern these flows and the mathematical models can be found in [2], [3], [4], [7]. However, the phenomenon of capillary trapping and its mathematical modelization have only been completed in some simplified cases [5], [9], [14].
The aim of this paper is to propose a general model for this phenomenon, and to give the mathematical study of the convergence of a scheme which can be used in the industrial context.

We thus consider an incompressible and immiscible oil-water flow through a 1D, 2D

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or 3D heterogeneous and isotropic porous medium Ω. Using Darcy's law, the conservation of oil and water phases is given, for all $(x,t) \in \Omega \times (0,T)$, by

$$
(1.1) \qquad \begin{cases} -\phi(x)\dfrac{\partial u(x,t)}{\partial t} - \operatorname{div}\Big(\mu_w(x,u(x,t))(\nabla p_w(x,t) - \rho_w g)\Big) = 0, \\ \phi(x)\dfrac{\partial u(x,t)}{\partial t} - \operatorname{div}\Big(\mu_o(x,u(x,t))(\nabla p_o(x,t) - \rho_o g)\Big) = 0, \\ p_o(x,t) - p_w(x,t) = \pi(x,u(x,t)) \end{cases}
$$

where the function ϕ is the porosity of the medium, $u \in [0,1]$ is the oil saturation (and therefore $1-u$ is the water saturation), $\pi(x,u)$ is the capillary pressure, g is the gravity acceleration. The indices o and w respectively stand for the oil and the water phase. Thus, for $\beta = o, w$, p_β is the pressure of the phase β, $\mu_\beta(x,u)$ is the mobility of the phase β and ρ_β is the density of the phase β. The unknowns of the problem are the functions u, p_w and p_o.

Focusing on the modeling of flow at the interface between two different porous materials, we make the following assumptions.

Assumptions 1.1.

H1-1. *The domain Ω is such that $\Omega = \Omega_1 \bigcup \Omega_2$. The subdomains Ω_1 and Ω_2 are disjoint open segments (if $d = 1$), polygonal (if $d = 2$) or polyhedral (if $d = 3$) bounded connected subsets of $\mathbb{R}^d$. We assume that the common boundary between Ω_1 and Ω_2, $\Gamma = \partial\overline{\Omega}_1 \bigcap \partial\overline{\Omega}_2$, has a strictly positive and finite $d-1$-measure. The real $T > 0$ is the length of the considered time period.*

H1-2. *The function ϕ takes the strictly positive constant value $0 < \phi_i < 1$ in Ω_i, for $i = 1, 2$.*

H1-3. *For $\beta \in \{o, w\}$, $i = 1, 2$ and for all $x \in \Omega_i$ $\mu_\beta(x,.) = \mu_{\beta,i}$. $\mu_{o,i}$ is a strictly increasing continuous function satisfying $\mu_{o,i}(u) = \mu_{o,i}(0) = 0$ for all $u \le 0$ and $\mu_{o,i}(u) = \mu_{o,i}(1)$ for all $u \ge 1$. $\mu_{w,i}$ is a strictly decreasing continuous function satisfying $\mu_{w,i}(u) = \mu_{w,i}(1) = 0$ for all $u \ge 1$ and $\mu_{w,i}(u) = \mu_{w,i}(0)$ for all $u \le 0$.*

H1-4. *For all $x \in \Omega_i$, $\pi(x,.) = \pi_i \in C^0(\mathbb{R},\mathbb{R})$ and π_i is such that its restriction $\pi_{i|[0,1]}$ to $[0,1]$ is strictly increasing, belongs to $C^1([0,1],\mathbb{R})$ and satisfies $\pi_i(u) = \pi_i(0)$ for all $u \le 0$ and $\pi_i(u) = \pi_i(1)$ for all $u \ge 1$. We assume that $\pi_1(0) \le \pi_2(0) \le \pi_1(1) \le \pi_2(1)$. We denote by $u_1^\star$ the unique real in $[0,1]$ satisfying $\pi_1(u_1^\star) = \pi_2(0)$. Thus, for all $u \in [0, u_1^\star)$, we have $\pi_1(u) < \pi_2(u)$. We denote by $u_2^\star$ the unique real in $[0,1]$ satisfying $\pi_2(u_2^\star) = \pi_1(1)$. Thus, for all $u \in (u_2^\star, 1]$, we have $\pi_1(u) < \pi_2(u)$. (See Figure 1.1.)*

H1-5. *The initial condition in saturation $u_{\rm ini} \in L^\infty(\Omega)$ and $0 \le u_{\rm ini}(x) \le 1$, for a.e. $x \in \Omega$.*

The following conditions must be satisfied on the traces of u_i, $p_{\beta,i}$ and $\nabla p_{\beta,i}$ on $\Gamma \times (0,T)$, respectively denoted by $u_{i,\Gamma}$, $p_{\beta,i,\Gamma}$ and $(\nabla p)_{\beta,i,\Gamma}$ (see [3]):

1. for any $\beta = o, w$, the flux of the phase β must be continuous:

$$
(1.2) \quad \mu_{\beta,1}(u_{1,\Gamma})((\nabla p)_{\beta,1,\Gamma} - \rho_\beta g).\overrightarrow{n}_{1,\Gamma} = -\mu_{\beta,2}(u_{2,\Gamma})((\nabla p)_{\beta,2,\Gamma} - \rho_\beta g).\overrightarrow{n}_{2,\Gamma}
$$

   where $\overrightarrow{n}_{i,\Gamma}$ is the unit normal of Γ outward to Ω_i,

2. for any $\beta = o, w$, either (p_β is continuous) or (p_β is discontinuous and $\mu_\beta = 0$); since the saturation is itself discontinuous across Γ, one must express the mobility at the upstream side of the interface. This gives

$$
(1.3) \qquad \mu_{\beta,1}(u_{1,\Gamma})(p_{\beta,1,\Gamma} - p_{\beta,2,\Gamma})^+ - \mu_{\beta,2}(u_{2,\Gamma})(p_{\beta,2,\Gamma} - p_{\beta,1,\Gamma})^+ = 0
$$

along with $p_{o,i,\Gamma} - p_{w,i,\Gamma} = \pi_i(u_{i,\Gamma})$, for $i = 1, 2$, where we denote, for all $a \in \mathbb{R}$, $a^+ = \max(a, 0)$.

The relations (1.3) can be directly expressed in terms of relations between $u_{i,\Gamma}$ and $p_{\beta,i,\Gamma}$, $\beta = o, w$, $i = 1, 2$:

1. If $0 \le u_{1,\Gamma} < u_1^\star$, then $\mu_{w,1}(u_{1,\Gamma}) > 0$; this implies $p_{w,1,\Gamma} \le p_{w,2,\Gamma}$. Since $\pi_1(u_{1,\Gamma}) < \pi_2(0) \le \pi_2(u_{2,\Gamma})$, we get $p_{o,1,\Gamma} < p_{o,2,\Gamma}$, which in turn implies $\mu_{o,2}(u_{2,\Gamma}) = 0$, and thus $u_{2,\Gamma} = 0$. Therefore $\mu_{w,2}(u_{2,\Gamma}) > 0$, and $p_{w,2,\Gamma} \le p_{w,1,\Gamma}$. Thus $p_{w,2,\Gamma} = p_{w,1,\Gamma}$. In this case, the oil phase is trapped in Ω_1, and the water flows across Γ.
2. If $u_1^\star \le u_{1,\Gamma}$ and $u_{2,\Gamma} \le u_2^\star$, then $\pi_2(0) \le \pi_1(u_{1,\Gamma})$, and $\pi_2(u_{2,\Gamma}) \le \pi_1(1)$. Since $\mu_{o,1}(u_{1,\Gamma}) > 0$, then $p_{o,1,\Gamma} \le p_{o,2,\Gamma}$ and $\mu_{o,2}(u_{2,\Gamma}) = 0$ or $p_{o,1,\Gamma} = p_{o,2,\Gamma}$. Similarly, since $\mu_{w,2}(u_{2,\Gamma}) > 0$, then $p_{w,1,\Gamma} \ge p_{w,2,\Gamma}$ and $\mu_{w,1}(u_{1,\Gamma}) = 0$ or $p_{w,1,\Gamma} = p_{w,2,\Gamma}$. Therefore, we get $p_{o,1,\Gamma} - p_{w,1,\Gamma} \le p_{o,2,\Gamma} - p_{w,2,\Gamma}$, which gives $\pi_1(u_{1,\Gamma}) \le \pi_2(u_{2,\Gamma})$. If we consider the case $\mu_{o,2}(u_{2,\Gamma}) = 0$, we get $u_{2,\Gamma} = 0$ and thus $\pi_2(0) = \pi_1(u_{1,\Gamma})$. Similarly, if we consider the case $\mu_{w,1}(u_{1,\Gamma}) = 0$, we get $\pi_2(u_{2,\Gamma}) = \pi_1(1)$. If we have at the same time $\mu_{o,2}(u_{2,\Gamma}) > 0$ and $\mu_{w,1}(u_{1,\Gamma}) > 0$, then $p_{o,1,\Gamma} = p_{o,2,\Gamma}$ and $p_{w,1,\Gamma} = p_{w,2,\Gamma}$, which implies $\pi_1(u_{1,\Gamma}) = \pi_2(u_{2,\Gamma})$. Therefore, in all cases, we get $\pi_1(u_{1,\Gamma}) = \pi_2(u_{2,\Gamma})$, and consequently $p_{o,1,\Gamma} = p_{o,2,\Gamma}$ and $p_{w,1,\Gamma} = p_{w,2,\Gamma}$. In this case, both phases flow across Γ.
3. If $u_2^\star < u_{2,\Gamma} \le 1$, a similar discussion yields $u_{1,\Gamma} = 1$ and $p_{o,1,\Gamma} = p_{o,2,\Gamma}$. In this case, the water phase is trapped in Ω_1, and the oil flows across Γ.

A consequence of this discussion is that, in all cases, the resulting condition on the oil saturations at the boundary Γ is given by $\hat\pi_1(u_{1,\Gamma}) = \hat\pi_2(u_{2,\Gamma})$, defining the functions $\hat\pi_1$ and $\hat\pi_2$ by $\hat\pi_1 : u \mapsto \max(\pi_1(u), \pi_2(0))$ and $\hat\pi_2 : u \mapsto \min(\pi_2(u), \pi_1(1))$.

Now let us introduce the global pressure $\tilde p_i(x,t) = p_{w,i}(x,t) + \displaystyle\int_0^{u_i(x,t)} \frac{\mu_{o,i}(a)}{\mu_{o,i}(a) + \mu_{w,i}(a)} \pi_i'(a)da$ (first introduced by Chavent, see for example [7]) and the functions $\eta_i : u \mapsto \dfrac{\mu_{o,i}(u)\mu_{w,i}(u)}{\mu_{o,i}(u) + \mu_{w,i}(u)}$ and $\varphi_i : u \mapsto \displaystyle\int_0^u \eta_i(a)\pi_i'(a)da$. We denote by L_{φ_i} the Lipschitz constant of φ_i and by C_η an upper bound of $\eta_i(u)$, $u \in \mathbb{R}$, $i = 1$ and 2. Using these notations we have for $(x,t) \in \Omega_i \times (0,T)$, $i = 1, 2$,

$$(1.4)\quad \left\{ \begin{array}{l} \phi_i \dfrac{\partial u_i(x,t)}{\partial t} - \operatorname{div}\Big(\mu_{o,i}(u_i(x,t))(\nabla \tilde p_i(x,t) - \rho_o g)\Big) - \Delta\varphi_i(u_i(x,t)) = 0, \\ -\operatorname{div}\left(\displaystyle\sum_{\beta=o,w} \mu_{\beta,i}(u_i(x,t))\nabla\tilde p_i(x,t) - \sum_{\beta=o,w} \mu_{\beta,i}(u_i(x,t))\rho_\beta g \right) = 0. \end{array} \right.$$

We neglect in the first equation of (1.4) the term $\operatorname{div}\left[\mu_{o,i}(u_i(x,t))(\nabla\tilde p_i(x,t) - \rho_o g)\right]$ in front of $\Delta\varphi_i(u_i(x,t))$, since this is sufficient to get the mathematical properties which are involved in the oil trapping phenomenon, as shown in the numerical examples at the end of this paper. Equations (1.2), (1.3) and (1.4) then produce within this simplified case the following equations, the solution of which are the functions $u_i(x,t)$, $(x,t) \in \Omega_i \times (0,T)$:

$$(1.5)\qquad \phi_i \frac{\partial u_i}{\partial t} - \Delta\varphi_i(u_i) = 0, \text{in } \Omega_i \times (0,T), \text{for all } i \in \{1,2\},$$

$$\nabla\varphi_1(u_{1,\Gamma}).\vec{n}_{1,\Gamma} = -\nabla\varphi_2(u_{2,\Gamma}).\vec{n}_{2,\Gamma}, \text{ on } \Gamma\times(0,T) \tag{1.6}$$

and

$$\hat{\pi}_1(u_{1,\Gamma}) = \hat{\pi}_2(u_{2,\Gamma}), \tag{1.7}$$

which summarizes the discussion induced by (1.3). Considering the problem of the migration of oil, we prescribe a homogeneous Neumann condition, which is expressed by

$$\eta(.,u)\nabla\pi(.,u).\vec{n} = 0, \text{ on } \partial\Omega\times(0,T). \tag{1.8}$$

For $t = 0$, we have

$$u(x,0) = u_{\text{ini}}, \text{ in } \Omega. \tag{1.9}$$

Before giving the weak formulation of the problem we prove the following Lemma.

LEMMA 1.2. *Under Assumptions 1.1, let* $\Psi : [\pi_2(0),\pi_1(1)] \to \mathbb{R}$ *be the strictly increasing function defined by* $p \mapsto \Psi(p) = \int_{\pi_2(0)}^{p} \min(\eta_1(\pi_1^{(-1)}(a)), \eta_2(\pi_2^{(-1)}(a)))da$. *For all* $i \in \{1,2\}$, *the function* $\Psi\circ\hat{\pi}_i\circ\varphi_i^{(-1)}$ *is Lipschitz continuous with a constant lower than* 1.

Proof. For $i = 1$ or 2, let a be real such that $\varphi_1(u_1^\star) < a < \varphi_1(1)$ if $i = 1$, $0 < a < \varphi_2(u_2^\star)$ if $i = 2$. Within such a condition, we have $\hat{\pi}_i(\varphi_i^{(-1)}(a)) = \pi_i(\varphi_i^{(-1)}(a))$. Let us calculate the derivative of the function $\pi_i\circ\varphi_i^{(-1)}$. Let $b \neq a$ be a real such that $\varphi_1(u_1^\star) < b < \varphi_1(1)$ if $i = 1$, $0 < b < \varphi_2(u_2^\star)$ if $i = 2$; setting $A = \varphi_i^{(-1)}(a)$ and $B = \varphi_i^{(-1)}(b)$, we have

$$\frac{\pi_i(\varphi_i^{(-1)}(b)) - \pi_i(\varphi_i^{(-1)}(a))}{b-a} = \frac{\pi_i(B)-\pi_i(A)}{\varphi_i(B)-\varphi_i(A)}.$$

Let us denote by $I(A,B)$ the interval $[A,B]$ if $B \geq A$, $[B,A]$ otherwise. Using the definition of φ_i, we have

$$\left(\min_{C\in I(A,B)} \eta_i(C)\right)(\pi_i(B)-\pi_i(A)) \leq \varphi_i(B)-\varphi_i(A) \leq \\ \left(\max_{C\in I(A,B)} \eta_i(C)\right)(\pi_i(B)-\pi_i(A)),$$

and therefore there exists $C \in I(A,B)$ such that $\varphi_i(B)-\varphi_i(A) = \eta_i(C)(\pi_i(B) - \pi_i(A))$. Thus

$$\frac{\pi_i(\varphi_i^{(-1)}(b)) - \pi_i(\varphi_i^{(-1)}(a))}{b-a} = \frac{1}{\eta_i(C)},$$

which gives, letting $b\to a$, $(\pi_i\circ\varphi_i^{(-1)})'(a) = \frac{1}{\eta_i(\varphi_i^{(-1)}(a))}$. We thus get that the function $\Psi\circ\hat{\pi}_i\circ\varphi_i^{(-1)}$ has a derivative in a which is

$$(\Psi\circ\hat{\pi}_i\circ\varphi_i^{(-1)})'(a) = \Psi'(\pi_i(\varphi_i^{(-1)}(a)))(\pi_i\circ\varphi_i^{(-1)})'(a) = \frac{\Psi'(\pi_i(\varphi_i^{(-1)}(a)))}{\eta_i(\varphi_i^{(-1)}(a))}.$$

Using the definition of Ψ, we get $\Psi'(\pi_i(y)) \leq \eta_i(y)$ for $y = \varphi_i^{(-1)}(a)$. Gathering these results, we get that

$$(\Psi \circ \hat{\pi}_i \circ \varphi_i^{(-1)})'(a) \leq 1.$$

If $i = 1$ and $0 < a < \varphi_1(u_1^\star)$, or if $i = 2$ and $\varphi_2(u_2^\star) < a < 1$, then the function $\Psi \circ \hat{\pi}_i \circ \varphi_i^{(-1)}$ is constant, which implies a zero derivative. This completes the proof of the lemma. ◻

The system (1.5)–(1.9) is a nonlinear parabolic problem defined on a heterogeneous domain. Since in the general case, such a problem does not have any strong solution, we now give the definition of a weak solution to this problem.

Definition 1.3. *Under Assumptions 1.1, a weak solution u of the problem (1.5)–(1.9) is defined by*

1. *for all $i \in \{1, 2\}$, $u = u_i$ in $\Omega_i \times (0, T)$ with*

$$u_i \in L^\infty(\Omega_i \times (0,T)),\ 0 \leq u_i \leq 1 \text{ a.e. and } \varphi_i(u_i) \in L^2(0,T; H^1(\Omega_i)),$$

2. *for all $\psi \in C_{test} = \{h \in H^1(\Omega \times (0,T)),\ \ h(.,T) = 0\}$,*

$$\sum_{i=1}^{2} \left[\begin{array}{l} \displaystyle\int_0^T \int_{\Omega_i} [\phi_i u_i(x,t)\psi_t(x,t) - \nabla\varphi_i(u_i(x,t)).\nabla\psi(x,t)]\ dxdt + \\ \displaystyle\int_{\Omega_i} \phi_i u_{\text{ini}}(x,0)\psi(x,0)dx \end{array} \right] = 0,$$

3. *the function $w : \Omega \times (0,T) \to \mathbb{R}$ defined by $(x,t) \mapsto \Psi(\hat{\pi}_i(u_i(x,t)))$ for a.e. $(x,t) \in \Omega_i \times (0,T)$, $i = 1, 2$, belongs to $L^2(0,T; H^1(\Omega))$.*

Remark 1.4. This weak formulation is sufficient to impose (1.5),(1.6),(1.8),(1.9) on regular solutions. The last condition given in Definition 1.3 is a functional method to impose the condition (1.7).

In the homogeneous case, i.e. $\phi_1 = \phi_2$, $\pi_1 = \pi_2$ and $\eta_1 = \eta_2$, classical results of existence and uniqueness of a solution are available (see for instance [1] and [6] for a uniqueness result in more general cases). A simplified case of (1.5)–(1.9) has been handled in the heterogeneous case in [5], where the authors handle the case $d = 1$, $\Omega_1 = (-\infty, 0)$, $\Omega_2 = (0, +\infty)$, and for $i = 1, 2$, $\phi_i = 1$, $\eta_i(u) = k_i u$ and $\pi_i(u) = (1 + u)/\sqrt{k_i}$, where $0 < k_2 < k_1$ (note that only the problem of the oil trapping is considered here, since the physical conditions $\eta_i(1) = 0$ is not ensured). Under additional hypotheses of regularity on the initial data, the authors get the existence and the uniqueness of the solution to the problem (1.5)–(1.9). We focus in this paper on the convergence of a numerical scheme for the approximation of u, in the general framework of Assumptions 1.1. Up to a subsequence, we prove (see Theorem 2.15) the convergence of the finite volume scheme given by the equations (2.2)–(2.4) to a weak solution in the sense of Definition 1.3. As an immediate consequence, the convergence of the scheme gives the existence of a solution to the problem (1.5)–(1.9) (see Corollary 2.17). Similar works have already been done for example in [12], [13] in the case of a homogeneous domain. Therefore, in the following proofs, we only insist on the new elements which appear in our study, mainly related to the presence of two domains linked by the equations (1.6)-(1.7) (or (2.4) for the discrete problem). We end up this study with numerical results (see §3) and concluding remarks on ongoing works and future prospects (see §4).

2. Study of a finite volume scheme. In this section, we study a finite volume scheme discretizing the equations (1.5)–(1.9). First we define an admissible discretization of $\Omega \times (0,T)$.

2.1. Admissible discretization of $\Omega \times (0,T)$.

Definition 2.1 (Admissible mesh). *We denote by $\mathcal{M}$ an admissible finite volume discretization on a domain Ω ; $\mathcal{M}$ is composed of a triplet $(\mathcal{T},\mathcal{E},\mathcal{P})$ with $\mathcal{T} = \mathcal{T}_1 \bigcup \mathcal{T}_2$, $\mathcal{E} = \mathcal{E}_1 \bigcup \mathcal{E}_2$ and $\mathcal{P} = \mathcal{P}_1 \bigcup \mathcal{P}_2$ which satisfy the following properties.*

- *For $i \in \{1,2\}$, $\mathcal{T}_i$ is a family of control volumes which are nonempty open polygonal convex disjoint subsets of Ω_i. These elements satisfy $\bigcup_{K\in\mathcal{T}_i} \overline{K} = \overline{\Omega}_i$.*
  We denote by $\partial K = \overline{K} \setminus K$ the boundary of volume K and by $m(K)$ its measure (its length for $d=1$, its area for $d=2$, its volume for $d=3$).
- *For $i \in \{1,2\}$, $\mathcal{E}_i$ stands for the set of the edges of the control volumes in $\mathcal{T}_i$. For all $\sigma \in \mathcal{E}_i$, there exist a hyperplane E of $\mathbb{R}^d$ and a control volume $K \in \mathcal{T}_i$ such that $\overline{\sigma} = E \bigcap \partial K$ and σ is a nonempty open subset of E. We denote by $\mathcal{E}_K$ the subset of $\mathcal{E}$ composed of the edges of the volume K. Then we have $\partial K = \bigcup_{\sigma\in\mathcal{E}_K} \overline{\sigma}$. For any $\sigma \in \mathcal{E}_i$, we have*
  - *either $\sigma \in \mathcal{E}_{int,i} = \{\sigma \in \mathcal{E}_i,\ \exists\ (K,L) \in \mathcal{T}_i^2,\ K \neq L$ such that $\overline{\sigma} = \overline{K} \bigcap \overline{L} \neq \emptyset\}$ (in that case σ is also denoted by $K|L$),*
  - *or $\sigma \in \mathcal{E}_\Gamma = \{\sigma \in \mathcal{E}_i,\ \exists\ (K,L) \in \mathcal{T}_1 \times \mathcal{T}_2,\ K \neq L$ such that $\overline{\sigma} = \overline{K} \bigcap \overline{L} \neq \emptyset\}$,*
  - *or $\sigma \in \mathcal{E}_{ext,i} = \{\sigma \in \mathcal{E}_i,\ \exists K \in \mathcal{T}_i$ such that $\bar{\sigma} = \partial K \bigcap (\partial\Omega_i \setminus \Gamma) \neq \emptyset\}$.*
- *For $i \in \{1,2\}$, $\mathcal{P}_i$ refers to a family of points $(x_K)_{K\in\mathcal{T}}$ satisfying the following properties:*
  - *$x_K \in K$,*
  - *for all $L \in \mathcal{T}_j$, $j \in \{1,2\}$, the straight line (x_K, x_L) going through x_K and x_L is orthogonal to $K|L$.*

  We also set
  - *$\mathcal{T}_\Gamma = \{(K,L),\ K \in \mathcal{T}_1,\ L \in \mathcal{T}_2,\ K|L \in \mathcal{E}_\Gamma\}$,*
  - *$\mathcal{E}_{int} = \mathcal{E}_{int,1} \bigcup \mathcal{E}_{int,2} \bigcup \mathcal{E}_\Gamma$,*
  - *$\mathcal{E}_{ext} = \mathcal{E}_{ext,1} \bigcup \mathcal{E}_{ext,2}$.*

  For $i = 1,2$, the set of the neighbouring volumes of a volume $K \in \mathcal{T}_i$ within Ω_i is represented by $N(K) = \{L \in \mathcal{T}_i,\ K|L \in \mathcal{E}_K\}$. The unit normal of an edge $K|L \in \mathcal{E}_{int}$ outward to K is denoted by $\overrightarrow{n}_{K,L}$. The area of an edge σ is denoted by $m(\sigma)$. For all $K \in \mathcal{T}$, $\sigma \in \mathcal{E}_K$, $d_{K,\sigma}$ stands for the euclidean distance between x_K and the edge σ and for $K|L \in \mathcal{E}_{int}$, $d_{K|L}$ is the euclidean distance between x_K and x_L. Using these notations the transmissivity $\tau_{K|L}$ through $K|L$ is equal to $\dfrac{m(K|L)}{d_{K|L}}$ and, for $\sigma \in \mathcal{E}_{ext}$ with $\sigma \in \mathcal{E}_K$, the transmissivity $\tau_{K,\sigma}$ through σ is equal to $\dfrac{m(\sigma)}{d_{K,\sigma}}$. For $i \in \{1,2\}$ and $K|L \in \mathcal{E}_{int,i}$, we denote by $D_{K|L}$ the union of the two cones with the respective vertices x_K and x_L and the basis $K|L$. For $\sigma \in \mathcal{E}_{ext}$ such that $\sigma \in \mathcal{E}_K$, D_σ is the cone with vertex x_K and basis σ.

We set $\mathrm{size}(\mathcal{M}) = \sup\{\mathrm{diam(K)}, \mathrm{K} \in \mathcal{T}\}$. *The regularity of the mesh is defined by*

$$\mathrm{regul}(\mathcal{M}) = \frac{\mathrm{size}(\mathcal{M})}{\min_{K\in\mathcal{T},\sigma\in\mathcal{E}_K} d_{K,\sigma}}. \tag{2.1}$$

In this paper, for the sake of simplicity, we restrict our study to constant time steps. But all results stated in the following can be adjusted to variable time steps.

Definition 2.2 (Admissible time discretization of $(0,T)$).

A discretization of $(0,T)$ is given by an integer $M \in \mathbb{N}$ such that $\delta t = \frac{T}{M+1}$. The increasing sequence of times $(t_n)_{n\in\{0\ldots M+1\}}$ which discretizes $(0,T)$ is then given by $t_n = n\delta t$.

Definition 2.3 (Admissible discretization of $\Omega \times (0,T)$).

An admissible discretization $\mathcal{D}$ of $\Omega \times (0,T)$ is composed of a pair $(\mathcal{M}, M)$ where $\mathcal{M}$ is an admissible discretization of Ω and $M \in \mathbb{N}$ (see Definitions 2.1 and 2.2). We then denote $\mathrm{size}(\mathcal{D}) = \max(\mathrm{size}(\mathcal{M}), \delta t)$.

2.2. Discrete functional properties. Let $\mathcal{D}$ be an admissible discretization of the domain $\Omega \times (0,T)$ (see Definition 2.3), $K \in \mathcal{T}$ and $n \in \{0\ldots M\}$. For a variable u, we denote by u_K^{n+1} its approximation over the volume K and over the time interval $]n\delta t, (n+1)\delta t]$ and by $(u_K^0)_{K\in\mathcal{T}}$ a piecewise constant approximation of the initial condition. We denote by

- $\mathcal{X}(\mathcal{T})$ the set of piecewise constant functions over the mesh $\mathcal{T}$: $u_\mathcal{T} \in \mathcal{X}(\mathcal{T})$ is defined, for all $x \in \Omega$, by $u_\mathcal{T}(x) = u_K$ for $x \in K$,
- $\mathcal{X}(\mathcal{D})$ the set of piecewise constant functions over the discretization $\mathcal{D}$: $u_\mathcal{D} \in \mathcal{X}(\mathcal{D})$ is defined, for all $n \in \{0\ldots M\}$, by $u_\mathcal{D}(.,t) = u_\mathcal{T}^{n+1} \in \mathcal{X}(\mathcal{T})$ for $t \in]n\delta t, (n+1)\delta t]$.

We introduce the notation $\delta u_{K,L} = u_L - u_K$.

For $i \in \{1,2\}$, the discrete $L^2(0,T;H^1(\Omega_i))$-seminorm is defined as follows:

Definition 2.4. *Let $\Omega \times (0,T)$ be a domain satisfying* H1-1 *and $\mathcal{D}$ be an admissible discretization of this domain in the sense of Definition 2.3. For $i \in \{1,2\}$, the $L^2(0,T;H^1(\Omega_i))$-seminorm of a function $u_\mathcal{D} \in \mathcal{X}(\mathcal{D})$ is defined by*

$$|u_\mathcal{D}|^2_{1,\mathcal{D},i} = \sum_{n=0}^{M} \delta t \sum_{K|L\in\mathcal{E}_{int,i}} \tau_{K|L} (\delta u_{K,L}^{n+1})^2.$$

2.3. An implicit scheme. The initial condition u_K^0 is given by

$$u_K^0 = \frac{1}{m(K)} \int_K u_{\mathrm{ini}}(x)\, dx, \ \forall K \in \mathcal{T}. \tag{2.2}$$

For the following time steps, $n \in \{0,\ldots,M\}$, we compute a discrete solution in saturation $(u_K^{n+1})_{K\in\mathcal{T}}$ thanks to the scheme

$$\begin{aligned} &m(K)\phi_i \frac{u_K^{n+1} - u_K^n}{\delta t} + \sum_{L\in N(K)} \tau_{K|L} \left(\varphi_i(u_K^{n+1}) - \varphi_i(u_L^{n+1})\right) + \\ &\sum_{\sigma\in\mathcal{E}_\Gamma \bigcap \mathcal{E}_K} \tau_{K,\sigma} \left(\varphi_i(u_K^{n+1}) - \varphi_i(u_{K,\sigma}^{n+1})\right) = 0, \ K \in \mathcal{T}_i, \ i \in \{1,2\} \end{aligned} \tag{2.3}$$

where, for all $(K,L) \in \mathcal{T}_\Gamma$, and for given values of u_K^{n+1} and u_L^{n+1}, the values $u_{K,K|L}^{n+1}, u_{L,K|L}^{n+1} \in [0,1]$ are the unique solutions (according to Lemma 2.5 below) of the system

$$\left\{ \begin{array}{rcl} \tau_{K,K|L}(\varphi_1(u_K^{n+1}) - \varphi_1(u_{K,\sigma}^{n+1})) &=& \tau_{L,K|L}(\varphi_2(u_{L,\sigma}^{n+1}) - \varphi_2(u_L^{n+1})), \\ \hat{\pi}_1(u_{K,\sigma}^{n+1}) &=& \hat{\pi}_2(u_{L,\sigma}^{n+1}). \end{array} \right. \tag{2.4}$$

Lemma 2.5. *Under Assumptions 1.1, let $\alpha_i > 0$ be given for $i = 1, 2$. Let $(a, b) \in \mathbb{R}^2$. Then there exists one and only one pair $(c, d) \in [0, 1]^2$ such that*

$$\alpha_1(\varphi_1(a) - \varphi_1(c)) = \alpha_2(\varphi_2(d) - \varphi_2(b))$$

and

$$\hat{\pi}_1(c) = \hat{\pi}_2(d).$$

We then denote $c = U_1(a, b, \alpha_1, \alpha_2)$ and $d = U_2(a, b, \alpha_1, \alpha_2)$. Then the functions U_1 and U_2 are continuous and nondecreasing with respect to a and b. Moreover, the following inequalities hold

$$\begin{aligned} &0 \le (\varphi_1(a) - \varphi_1(c))(\pi_1(a) - \pi_1(c)) \le (\varphi_1(a) - \varphi_1(c))(\pi_1(a) - \pi_2(b)), \\ &0 \le (\varphi_2(d) - \varphi_2(b))(\pi_2(d) - \pi_2(b)) \le (\varphi_2(d) - \varphi_2(c))(\pi_1(a) - \pi_2(b)). \end{aligned} \tag{2.5}$$

Proof. Let us take as unknowns the values $C = \varphi_1(c)$ and $D = \varphi_2(d)$ and let us denote $A = \varphi_1(a)$ and $B = \varphi_2(b)$. Then (C, D) is solution of

$$\alpha_1 C + \alpha_2 D = \alpha_1 A + \alpha_2 B, \tag{2.6}$$

$$\hat{\pi}_1(\varphi_1^{(-1)}(C)) = \hat{\pi}_2(\varphi_2^{(-1)}(D)). \tag{2.7}$$

Let us first consider the case where $\alpha_1 A + \alpha_2 B \le \alpha_1 \varphi_1(u_1^\star)$. Since this implies $C \le \varphi_1(u_1^\star)$, we have necessarily $D = 0$ according to (2.7). Thus the solution is obtained, taking $D = 0$ and $C = (\alpha_1 A + \alpha_2 B)/\alpha_1$. In this case, since $D \le B$, we have $C \ge A$, and since $\pi_2(b) \ge \pi_2(0) \ge \pi_1(c) \ge \pi_1(a)$, we get (2.5).
We now consider the case where $\alpha_1\varphi_1(u_1^\star) < \alpha_1 A + \alpha_2 B < \alpha_1\varphi_1(1) + \alpha_2\varphi_2(u_2^\star)$. Since in this case we necessarily have $\varphi_1(u_1^\star) < C$ and $D < \varphi_2(u_2^\star)$ (see (2.7)), the relation $C = \varphi_1(\pi_1^{(-1)}(\pi_2(\varphi_2^{(-1)}(D))))$ holds, and since the function $D \mapsto \alpha_1\varphi_1(\pi_1^{(-1)}(\pi_2(\varphi_2^{(-1)}(D)))) + \alpha_2 D$ is continuous and strictly increasing, the system has one and only one solution (C, D). We then get in this case that $\pi_1(c) = \pi_2(d)$, and since $\pi_1(a) - \pi_1(c)$ has the same sign as $\pi_2(d) - \pi_2(b)$, we get (2.5).
Finally, the case $\alpha_1\varphi_1(1) + \alpha_2\varphi_2(u_2^\star) \le \alpha_1 A + \alpha_2 B$ is symmetric with the first case, and we get $C = \varphi_1(1)$ and $D = (\alpha_1(A - \varphi_1(1)) + \alpha_2 B)/\alpha_2$. We then have in this case $C \ge A$ and thus $D \le B$, and since $\pi_2(b) \ge \pi_2(d) \ge \pi_1(1) \ge \pi_1(a)$, we again get (2.5). In all these cases, C and D have been expressed as continuous nondecreasing functions of A and B, so the same conclusion holds for c and d as functions of a and b. ◻

Remark 2.6. It is possible to show that C and D, seen as functions of $A = \varphi_1(a)$ and $B = \varphi_2(b)$ verify, for a.e. $(a, b) \in \mathbb{R}^2$,

$$0 \le \frac{\partial C}{\partial A} \le 1,\ 0 \le \frac{\partial D}{\partial A} \le \frac{\alpha_1}{\alpha_2},\ 0 \le \frac{\partial C}{\partial B} \le \frac{\alpha_2}{\alpha_1} \text{ and } 0 \le \frac{\partial D}{\partial B} \le 1.$$

Now we can state the L^∞-stability of the scheme and then the existence of a solution to the equations (2.2)–(2.4).

2.4. L^∞-stability of the scheme. If Ω were a homogeneous porous medium we could prove that the discrete solution in saturation satisfies a maximum principle depending on the initial condition [12]. Here, in presence of a heterogeneity, this result does not hold any more.

PROPOSITION 2.7. *Under Assumptions 1.1, let $\mathcal{D}$ be an admissible discretization of the domain $\Omega \times (0,T)$ (see Definition 2.3) and $u_{\mathcal{T}}^{n+1} \in \mathcal{X}(\mathcal{T})$, $n \in \{0 \dots M\}$, the solution to the system (2.2)–(2.4) (the existence and uniqueness of such a solution is shown in Proposition 2.8). Then $u_{\mathcal{T}}^{n+1}$ satisfies*

$$\forall K \in \mathcal{T},\ 0 \le u_K^{n+1} \le 1. \tag{2.8}$$

Proof. For all $K \in \mathcal{T}_i$, $i \in \{1,2\}$, equations (2.2)–(2.4) imply

$$u_K^{n+1} = H_K(u_K^n, (u_L^{n+1})_{L \in \mathcal{T}})$$

with

$$H_K(a,(a_L)_{L\in\mathcal{T}}) = \frac{1}{1+\lambda_K}\Bigg(a + \lambda_K a_K + \frac{\delta t}{m(K)\phi_i}\left(\begin{array}{l}\displaystyle\sum_{L\in N(K)} \tau_{K|L}\left(\varphi_i(a_L) - \varphi_i(a_K)\right) + \\ \displaystyle\sum_{\sigma\in\mathcal{E}_\Gamma \bigcap \mathcal{E}_K} \tau_{K,\sigma}\left(\varphi_i(a_{K,\sigma}) - \varphi_i(a_K)\right)\end{array}\right)\Bigg),$$

and

$$\lambda_K = \frac{\delta t L_\varphi}{m(K)\phi_i}\left(\sum_{L\in N(K)} \tau_{K|L} + \sum_{\sigma\in\mathcal{E}_\Gamma \bigcap \mathcal{E}_K} \tau_{K,\sigma}\right)$$

and where, for all $(K,L) \in \mathcal{T}_\Gamma$, $a_{K,K|L}$ is defined by
$a_{K,K|L} = U_1(a_K, a_L, \tau_{K,K|L}, \tau_{L,K|L})$ and $a_{L,K|L} = U_2(a_K, a_L, \tau_{K,K|L}, \tau_{L,K|L})$ (the functions U_1 and U_2 are defined in Lemma 2.5).
Lemma 2.5 implies that the function $H_K(a,(a_L)_{L\in\mathcal{T}})$ is nondecreasing with respect to a and to a_L for all $L \in \mathcal{T}$ (including the case $L = K$).
Let us prove the above proposition by induction on n. It is true for $n = 0$. We assume that is true for n, and that there is $K_{\max} \in \mathcal{T}$ such that $K_{\max} = \max_{K\in\mathcal{T}}(u_K^{n+1})$ and $u_{K_{\max}}^{n+1} > 1$. Using the monotony of the function $H_{K_{\max}}$, we have

$$1 < u_{K_{\max}}^{n+1} \le H_{K_{\max}}(1, (u_{K_{\max}}^{n+1})_{L\in\mathcal{T}}) = \frac{1 + \lambda_{K_{\max}} u_{K_{\max}}^{n+1}}{1 + \lambda_{K_{\max}}}.$$

We then get a contradiction with the existence of such a $K_{\max}$. In the same way, we prove that there is no $K_{\min} \in \mathcal{T}_i$ such that $K_{\min} = \min_{K\in\mathcal{T}}(u_K^{n+1})$ and $u_{K_{\min}}^{n+1} < 0$. ∎

2.5. Existence and uniqueness of a discrete solution.

PROPOSITION 2.8. *Under Assumptions 1.1, let $\mathcal{D}$ be an admissible discretization of the domain $\Omega \times (0,T)$ (see Definition 2.3). Then, for all $n \in \{0 \dots M\}$, there exists one and only one solution $u_{\mathcal{T}}^{n+1} \in \mathcal{X}(\mathcal{T})$ to the system (2.2)–(2.4).*

Proof. The system composed of the equations (2.2)–(2.4) can be seen as a system with unknowns $(u_K^{n+1})_{K\in\mathcal{T}}$ thanks to Lemma 2.5.
We set $N = \mathrm{card}(\mathcal{T})$ and we consider the application $\psi : \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$ defined by $((u_K)_{K\in\mathcal{T}}, \lambda) \mapsto (v_K)_{K\in\mathcal{T}}$ with, for all $K \in \mathcal{T}$,

$$\begin{aligned} v_K = \quad & m(K)\phi_i \frac{u_K - u_K^n}{\delta t} + \lambda \sum_{L\in N(K)} \tau_{K|L} \left(\varphi_i(u_K) - \varphi_i(u_L)\right) + \\ & \lambda \sum_{\sigma\in\mathcal{E}_\Gamma \bigcap \mathcal{E}_K} \tau_{K,\sigma} \left(\varphi_i(u_K) - \varphi_i(u_{K,\sigma})\right), \end{aligned}$$

where, for all $(K, L) \in \mathcal{T}_\Gamma$, we take $u_{K,K|L} = U_1(u_K, u_L, \tau_{K,K|L}, \tau_{L,K|L})$ and $u_{L,K|L} = U_2(u_K, u_L, \tau_{K,K|L}, \tau_{L,K|L})$ (the functions U_1 and U_2 are defined in Lemma 2.5).
The function ψ is continuous with respect to each one of its arguments. Moreover, reproducing the proof of the Proposition 2.7 we can prove that, for all $\lambda \in [0, 1]$, $\psi((u_K)_{K\in\mathcal{T}}, \lambda) = (0)_{K\in\mathcal{T}}$ implies $u_K \in [0, 1]$ for all $K \in \mathcal{T}$. Since $\psi((u_K)_{K\in\mathcal{T}}, 0)$ is linear, an argument based on the topological degree (see [11] and references therein) implies that $\psi((u_K)_{K\in\mathcal{T}}, 1) = (0)_{K\in\mathcal{T}}$ admits at least one solution.
Turning now to the proof of uniqueness, we assume that, for a given $n \in \{0 \dots M\}$, $(u_K)_{K\in\mathcal{T}}$ and $(\tilde{u}_K)_{K\in\mathcal{T}}$ are two solutions of (2.2)–(2.4). Using, for all $K \in \mathcal{T}$, the functions H_K defined in the proof of Proposition 2.7, we get that

$$\max(u_K, \tilde{u}_K) \le H_K(u_K^n, (\max(u_L, \tilde{u}_L))_{L\in\mathcal{T}})$$

and

$$\min(u_K, \tilde{u}_K) \ge H_K(u_K^n, (\min(u_L, \tilde{u}_L))_{L\in\mathcal{T}}).$$

If we multiply the above inequalities by $(1 + \lambda_K)m(K)\phi_i$, if we substract the second inequality from the first one, and if we sum the result over $K \in \mathcal{T}$, the exchange terms between all the pairs of neighbouring grid blocks and in particular the terms including λ_K vanish, and we obtain

$$\sum_{i=1,2} \sum_{K\in\mathcal{T}_i} m(K)\phi_i |u_K - \tilde{u}_K| \le 0,$$

which proves the uniqueness of the solution. □

2.6. Convergence. The remaining part of this section is devoted to the convergence proof of the scheme (2.2)–(2.4). The first step consists in obtaining some compactness properties for the sequence of approximated solutions. This will be done thanks to Kolmogorov's theorem. In particular this theorem requires that the space and time translates of the approximated solutions remain bounded.

2.6.1. Upper bound on the space translates.

PROPOSITION 2.9. *Under Assumptions 1.1, let $\mathcal{D}$ be an admissible discretization of the domain $\Omega \times (0, T)$ in the sense of Definition 2.3. Let $u_\mathcal{D} \in \mathcal{X}(\mathcal{D})$ be the solution of the equations (2.2)–(2.4). Then, there is $C_1 > 0$ only depending on η_j, π_j, Ω_j, $j \in \{1, 2\}$ such that*

$$\begin{aligned} 0 \le & \sum_{n=0}^{M} \delta t \sum_{(K,L)\in\mathcal{E}_\Gamma} \tau_{K,K|L} \left(\varphi_1(u_K^{n+1}) - \varphi_1(u_{K,K|L}^{n+1})\right) \left(\pi_1(u_K^{n+1}) - \pi_2(u_L^{n+1})\right) = \\ & \sum_{n=0}^{M} \delta t \sum_{(K,L)\in\mathcal{E}_\Gamma} \tau_{L,K|L} \left(\varphi_2(u_{L,K|L}^{n+1}) - \varphi_2(u_L^{n+1})\right) \left(\pi_1(u_K^{n+1}) - \pi_2(u_L^{n+1})\right) \le C_1 \end{aligned} \tag{2.9}$$

and, for $i \in \{1,2\}$, there exists $C_2 > 0$ depending on C_1 and on C_η such that

$$|\varphi_i(u_{\mathcal{D}})|^2_{1,\mathcal{D},i} \le C_2\,. \tag{2.10}$$

Proof. For $n \in \{0\dots M\}$ and $K \in \mathcal{T}_i$, we multiply the equation (2.3) by $\pi_i(u_K^{n+1})$ and we sum over the discretization $\mathcal{D}$. It leads to

$$\sum_{\substack{i=1\dots 2,\\ n=0\dots M,\\ K\in\mathcal{T}_i}} \left(\begin{array}{l} \left(m(K)\phi_i(u_K^{n+1}-u_K^n) + \delta t \left(\sum_{L\in N(K)} \tau_{K|L} \left(\varphi_i(u_K^{n+1}) - \varphi_i(u_L^{n+1}) \right) + \right.\right. \\ \left.\left. \sum_{\sigma\in\mathcal{E}_\Gamma\cap\mathcal{E}_K} \tau_{K,\sigma} \left(\varphi_i(u_K^{n+1}) - \varphi_i(u_{K,\sigma}^{n+1}) \right)\right)\right) \pi_i(u_K^{n+1}) \end{array} \right) = 0.$$

Accumulation term

Since the function $\pi_i(.)$ is nondecreasing, the function g_i defined by $g_i(u) = \int_0^u \pi_i(a)\,da$ is therefore convex. So we have

$$(u_K^{n+1}-u_K^n)\pi_i(u_K^{n+1}) \ge g_i(u_K^{n+1}) - g_i(u_K^n).$$

Thus we get

$$\sum_{n=0}^{M}\sum_{K\in\mathcal{T}_i} m(K)\phi_i(u_K^{n+1}-u_K^n)\pi_i(u_K^{n+1}) \ge \sum_{K\in\mathcal{T}_i} m(K)\phi_i(g_i(u_K^{M+1}) - g_i(u_K^0)).$$

Moreover we notice that

$$|\sum_{K\in\mathcal{T}_i} m(K)\phi_i(g_i(u_K^{M+1}) - g_i(u_K^0))| \le m(\Omega_i)\Big(\int_0^1 |\pi_i(a)|\,da\Big).$$

Diffusion term

As $\varphi_i(b)-\varphi_i(a) \le C_\eta \displaystyle\int_a^b \pi_i'(u)\,du$, we have

$$\sum_{n=0}^{M}\delta t \sum_{K|L\in\mathcal{E}_{int,i}} \tau_{K|L}\left(\varphi_i(u_K^{n+1})-\varphi_i(u_L^{n+1})\right)\left(\pi_i(u_K^{n+1})-\pi_i(u_L^{n+1})\right) \ge$$
$$\frac{1}{C_\eta}\sum_{n=0}^{M}\delta t \sum_{K|L\in\mathcal{E}_{int,i}} \tau_{K|L}\Big(\varphi_i(u_K^{n+1})-\varphi_i(u_L^{n+1})\Big)^2.$$

For $(K,L)\in\mathcal{T}_\Gamma$, we apply (2.5). This leads to

$$\tau_{K,\sigma}(\varphi_1(u_K^{n+1}) - \varphi_1(u_{K,\sigma}^{n+1}))(\pi_1(u_K^{n+1}) - \pi_2(u_L^{n+1})) \ge 0.$$

Finally, gathering the lower and upper bounds we obtained, we get

$$\sum_{i=1}^{2}|\varphi_i(u_{\mathcal{D}})|^2_{1,\mathcal{D},i} \le C_\eta \sum_{i=1}^{2} m(\Omega_i)\Big(\int_0^1 |\pi_i(a)|\,da\Big) = C_2$$

and

$$0 \le \sum_{n=0}^{M}\delta t \sum_{\sigma=K|L\in\mathcal{E}_\Gamma} \tau_{K,\sigma}\left(\varphi_1(u_K^{n+1})-\varphi_1(u_{K,\sigma}^{n+1})\right)\left(\pi_1(u_K^{n+1})-\pi_2(u_L^{n+1})\right) \le$$
$$\sum_{i=1}^{2} m(\Omega_i)\Big(\int_0^1 |\pi_i(a)|\,da\Big) = C_1\,,$$

which concludes the proof. □

We recall the following result, given in [11].

Lemma 2.10. *Under Assumptions 1.1, let $\mathcal{D}$ be an admissible discretization of the domain $\Omega \times (0,T)$ in the sense of Definition 2.3. Let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be given by the equations (2.2)–(2.4). Let $i = 1,2$ and $\xi \in \mathbb{R}^d$. We define the domain $\Omega_{i,\xi}$ by*

$$\Omega_{i,\xi} = \{x \in \Omega_i \;/\; [x, x+\xi] \subset \Omega_i\}.$$

Then the function $\varphi_i(u_{\mathcal{D}})$ satisfies

$$\begin{aligned}\int_0^T \int_{\Omega_{i,\xi}} |\varphi_i(u_{\mathcal{D}}(x+\xi,t) - \varphi_i(u_{\mathcal{D}}(x,t)|^2 dxdt \leq \\ |\xi|\Big(|\xi| + 2\mathrm{size}(\mathcal{M})\Big)|\varphi_i(u_{\mathcal{D}})|^2_{1,\mathcal{D},i}.\end{aligned} \tag{2.11}$$

This result produces the following proposition.

Proposition 2.11. *Under Assumptions 1.1, let $\mathcal{D}$ be an admissible discretization of the domain $\Omega \times (0,T)$ in the sense of Definition 2.3. Let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be given by the equations (2.2)–(2.4). Let $i = 1,2$ and ω_i be an open bounded subset of Ω_i with a regular boundary. We define the function $\varphi_{\mathcal{D},\omega_i}$ by $\varphi_{\mathcal{D},\omega_i}(x,t) = \varphi_i(u_{\mathcal{D}}(x,t))$ for a.e. $(x,t) \in \omega_i \times (0,T)$, $\varphi_{\mathcal{D},\omega_i}(x,t) = 0$ if $(x,t) \notin \omega_i \times (0,T)$. Then there exists $C_3 > 0$, only depending on T, η_j, π_j, Ω_j, $j \in \{1,2\}$ and of ω_i, such that*

$$\|\varphi_{\mathcal{D},\omega_i}(.+\xi,.) - \varphi_{\mathcal{D},\omega_i}\|^2_{L^2(\mathbb{R}^{d+1})} \leq C_3\,|\xi|\Big(|\xi|+1\Big),\ \forall \xi \in \mathbb{R}^d. \tag{2.12}$$

Proof. This result is a direct consequence of Proposition 2.9 and of Lemma 2.10 and of the fact that the measure of $\{x \in \omega_i,\ [x,x+\xi] \not\subset \omega_i\}$ is bounded by $C_{\omega_i}|\xi|$. □

2.6.2. Upper bound on the time translates.

Proposition 2.12. *Under Assumptions 1.1, let $\mathcal{D}$ be an admissible discretization of the domain $\Omega \times (0,T)$ in the sense of Definition 2.3. Let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be given by the equations (2.2)–(2.4). Let $i = 1,2$ and ω_i be an open bounded subset of Ω_i with a regular boundary. We define the function $\varphi_{\mathcal{D},\omega_i}$ by $\varphi_{\mathcal{D},\omega_i}(x,t) = \varphi_i(u_{\mathcal{D}}(x,t))$ for a.e. $(x,t) \in \omega_i \times (0,T)$, $\varphi_{\mathcal{D},\omega_i}(x,t) = 0$ if $(x,t) \notin \omega_i \times (0,T)$. Then there exists $C_4 > 0$, only depending on T, η_j, π_j, ϕ_j, Ω_j, $j \in \{1,2\}$ and of ω_i, such that, for* size$(\mathcal{M})$ *small enough,*

$$\int_{\mathbb{R}}\int_{\Omega}\Big(\varphi_{\mathcal{D},\omega_i}(x,t+\tau) - \varphi_{\mathcal{D},\omega_i}(x,t)\Big)^2 dxdt \leq C_4\,|\tau|,\ \forall \tau \in \mathbb{R}. \tag{2.13}$$

Proof. We suppose that $\tau \in (0,T)$ (the case $\tau < 0$ is deduced from $\tau > 0$ and the case $\tau > T$ is a consequence of an easy bound of $\int_{\mathbb{R}}\int_{\Omega}(\varphi_{\mathcal{D},\omega_i}(x,t+\tau) - \varphi_{\mathcal{D},\omega_i}(x,t))^2)dxdt$). Let $i = 1,2$ and let $\Theta_i \in C_c^\infty(\Omega_i,[0,1])$ be such that, for all $x \in \omega_i$, $\Theta_i(x) = 1$. We suppose that size$(\mathcal{M})$ is small enough so that Θ_i vanishes on all $K \in \mathcal{T}_i$ having edges on the boundary of Ω_i. For all $K \in \mathcal{T}_i$, we set $\Theta_{i,K} = \dfrac{1}{m(K)}\displaystyle\int_K \Theta_i(x)\,dx$. Since the function φ_i is Lipschitz continuous, we have

$$\int_0^{T-\tau}\int_{\Omega}\Theta_i(x)\phi_i\Big(\varphi_i(u_{\mathcal{D}}(x,t+\tau)) - \varphi_i(u_{\mathcal{D}}(x,t))\Big)^2 dxdt \leq L_\varphi \int_0^{T-\tau} A(t)\,dt$$

with

$$A(t) = \int_\Omega \Theta_i(x)\phi_i\Big(\varphi_i(u(x,t+\tau)) - \varphi_i(u(x,t))\Big)\Big(u(x,t+\tau) - u(x,t)\Big)\,dx.$$

Following the method used in [11], we first write $A(t)$ as

$$A(t) = \sum_{K\in\mathcal{T}_i}\left(m(K)\Theta_{i,K}\phi_i\Big(\varphi_i(u_K^{n_1(t)+1}) - \varphi_i(u_K^{n_0(t)+1})\Big)\sum_{n=0}^{M}\mathcal{X}_n(t,t+\tau)(u_K^{n+1}-u_K^n)\right)$$

where the indices $n_0(t)$ and $n_1(t)$ satisfy $n_0(t)\delta t < t \le (n_0(t)+1)\delta t$, $n_1(t)\delta t < t+\tau \le (n_1(t)+1)\delta t$, and the function $\mathcal{X}_n(a,b)$ is such that $\mathcal{X}_n(a,b) = 1$ if $a < b$ and $n\delta t \in [a,b[$, and $\mathcal{X}_n(a,b) = 0$ otherwise.
Using the definition of the scheme, we get

$$\begin{aligned} A(t) = & \sum_{K\in\mathcal{T}_i}\Bigg(\Theta_{i,K}\Big(\varphi_i(u_K^{n_1(t)+1}) - \varphi_i(u_K^{n_0(t)+1})\Big) \\ & \sum_{n=0}^{M}\mathcal{X}_n(t,t+\tau)\sum_{L\in N(K)}\delta t\,\tau_{K|L}\Big(\varphi_i(u_K^{n+1}) - \varphi_i(u_L^{n+1})\Big)\Bigg). \end{aligned}$$

Gathering the terms by edges leads to

$$\begin{aligned} A(t) = & \sum_{n=0}^{M}\delta t\,\mathcal{X}_n(t,t+\tau)\sum_{K|L\in\mathcal{E}_{int,i}}\tau_{K|L}\left[\begin{array}{c}\Theta_{i,K}\Big(\varphi_i(u_K^{n_1(t)+1}) - \varphi_i(u_K^{n_0(t)+1})\Big) - \\ \Theta_{i,L}\Big(\varphi_i(u_L^{n_1(t)+1}) - \varphi_i(u_L^{n_0(t)+1})\Big)\end{array}\right]\times \\ & \Big(\varphi_i(u_K^{n+1}) - \varphi_i(u_L^{n+1})\Big). \end{aligned}$$

Applying the equality $2(\Theta_{i,K}a - \Theta_{i,L}b) = (\Theta_{i,K}+\Theta_{i,L})(a-b) + (\Theta_{i,K}-\Theta_{i,L})(a+b)$ we get that

$$A(t) \le A_0(t) + A_1(t) + A_2(t)$$

with

$$\begin{aligned} A_0(t) = & \sum_{n=0}^{M}\delta t\,\mathcal{X}_n(t,t+\tau)\sum_{K|L\in\mathcal{E}_{int,i}}\tau_{K|L}\left|\varphi_i(u_K^{n_1(t)+1}) - \varphi_i(u_L^{n_1(t)+1})\right|\times \\ & \left|\varphi_i(u_K^{n+1}) - \varphi_i(u_L^{n+1})\right|, \end{aligned}$$

$$\begin{aligned} A_1(t) = & \sum_{n=0}^{M}\delta t\,\mathcal{X}_n(t,t+\tau)\sum_{K|L\in\mathcal{E}_{int,i}}\tau_{K|L}\left|\varphi_i(u_K^{n_0(t)+1}) - \varphi_i(u_L^{n_0(t)+1})\right|\times \\ & \left|\varphi_i(u_K^{n+1}) - \varphi_i(u_L^{n+1})\right| \end{aligned}$$

and

$$A_2(t) = \sum_{n=0}^{M}\delta t\,\mathcal{X}_n(t,t+\tau)\sum_{K|L\in\mathcal{E}_{int,i}}\tau_{K|L}L_\varphi\left|\Theta_{i,K} - \Theta_{i,L}\right|\,\left|\varphi_i(u_K^{n+1}) - \varphi_i(u_L^{n+1})\right|.$$

We then use Young's inequality, Proposition 2.9 and the regularity of the function Θ, to bound $A_0(t)$, $A_1(t)$ and $A_2(t)$ by a sum of terms under the form $\sum_{n=0}^{M} \delta t \mathcal{X}_n(t,t+\tau)a^n$, $\sum_{n=0}^{M} \delta t \mathcal{X}_n(t,t+\tau)a^{n_0(t)}$, and $\sum_{n=0}^{M} \delta t \mathcal{X}_n(t,t+\tau)a^{n_1(t)}$, such that $0 \le a^n$ for all $n = 0\dots,M$, and such that $\delta t \sum_{n=0}^{M} a^n$ is bounded independently on the discretization. We then use the properties

$\int_0^{T-\tau} \sum_{n=0}^{M} \delta t \mathcal{X}_n(t,t+\tau)a^n dt \le \tau \delta t \sum_{n=0}^{M} a^n$,

$\int_0^{T-\tau} \sum_{n=0}^{M} \delta t \mathcal{X}_n(t,t+\tau)a^{n_0(t)} dt \le \tau \delta t \sum_{n=0}^{M} a^n$ and

$\int_0^{T-\tau} \sum_{n=0}^{M} \delta t \mathcal{X}_n(t,t+\tau)a^{n_1(t)} dt \le \tau \delta t \sum_{n=0}^{M} a^n$, proven in [11]. ☐

2.6.3. Upper bound on the discrete $L^2(0,T;H^1(\Omega))$-semi-norm of the function $w_{\mathcal{D}}$. Let $u_{\mathcal{D}}$ be given by the equations (2.2)–(2.4). We consider $w_{\mathcal{D}}$ defined by $w_K^{n+1} = \Psi(\hat{\pi}_i(u_K^{n+1}))$, for all $i = 1,2$ and $K \in \mathcal{T}_i$. The following proposition states that the discrete $L^2(0,T;H^1(\Omega))$-semi-norm of the function $w_{\mathcal{D}}$ remains bounded. We first recall the definition of this semi-norm defined on the whole domain Ω.

Definition 2.13. *Let $\Omega\times(0,T)$ be a domain satisfying* H1-1 *and $\mathcal{D}$ be an admissible discretization of this domain in the sense of Definition 2.3. The $L^2(0,T;H^1(\Omega))$-semi-norm of a function $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ is defined by*

$$|u_{\mathcal{D}}|^2_{1,\mathcal{D}} = \sum_{n=0}^{M} \delta t \sum_{K|L\in\mathcal{E}_{int}} \tau_{K|L}(\delta u_{K,L}^{n+1})^2 = \sum_{i=1,2} |u_{\mathcal{D}}|^2_{1,\mathcal{D},i} + \sum_{n=0}^{M} \delta t \sum_{(K,L)\in\mathcal{T}_\Gamma} \tau_{K|L}(\delta u_{K,L}^{n+1})^2.$$

Proposition 2.14. *Under Assumptions 1.1, let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.3. Let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be the solution of the equations (2.2)–(2.4). Then, there exists $C_5 > 0$ only depending on η_j, π_j, Ω_j, $j \in \{1,2\}$ such that*

$$|w_{\mathcal{D}}|^2_{1,\mathcal{D}} \le C_5\,. \tag{2.14}$$

Proof. For $K \in \mathcal{T}_i$ and $L \in N(K)$, using the property of Lipschitz continuity of $\Psi\circ\hat{\pi}_i\circ\varphi_i^{(-1)}$ (see Lemma 1.2), we get

$$(w_K^{n+1} - w_L^{n+1})^2 \le (\varphi_i(u_K^{n+1}) - \varphi_i(u_L^{n+1}))^2$$

and therefore, we deduce from (2.10)

$$|w_{\mathcal{D}}|^2_{1,\mathcal{D},i} \le C_2\,.$$

We now consider the case $(K,L) \in \mathcal{T}_\Gamma$. We have, since $\hat{\pi}_1(u_{K,K|L}^{n+1}) = \hat{\pi}_2(u_{L,K|L}^{n+1})$,

$$\begin{aligned}\tau_{K|L}(\Psi(\hat{\pi}_1(u_K^{n+1})) - \Psi(\hat{\pi}_2(u_L^{n+1})))^2 \le\ & \tau_{K,K|L}(\Psi(\hat{\pi}_1(u_K^{n+1})) - \Psi(\hat{\pi}_1(u_{K,K|L}^{n+1})))^2\\ & +\tau_{L,K|L}(\Psi(\hat{\pi}_2(u_{L,K|L}^{n+1})) - \Psi(\hat{\pi}_2(u_L^{n+1})))^2,\end{aligned}$$

thanks to the convexity of the function $x \mapsto x^2$ and to $1/\tau_{K|L} = 1/\tau_{K,K|L} + 1/\tau_{L,K|L}$. We again use the properties of $\Psi\circ\hat{\pi}_i\circ\varphi_i^{(-1)}$ (see Lemma 1.2):

$$(\Psi(\hat{\pi}_1(u_K^{n+1})) - \Psi(\hat{\pi}_1(u_{K,K|L}^{n+1})))^2 \le (\varphi_1(u_K^{n+1}) - \varphi_1(u_{K,K|L}^{n+1}))^2,$$

and

$$(\Psi(\hat{\pi}_2(u_{L,K|L}^{n+1})) - \Psi(\hat{\pi}_2(u_L^{n+1})))^2 \le (\varphi_2(u_L^{n+1}) - \varphi_2(u_{L,K|L}^{n+1}))^2.$$

Now, using (2.5), we have, for all $(K,L) \in \mathcal{T}_\Gamma$,

$$
\begin{aligned}
&\Big(\varphi_1(u_K^{n+1}) - \varphi_1(u_{K,K|L}^{n+1})\Big)^2 \leq \\
&\Big(\varphi_1(u_K^{n+1}) - \varphi_1(u_{K,K|L}^{n+1})\Big) C_\eta \Big(\pi_1(u_K^{n+1}) - \pi_1(u_{K,K|L}^{n+1})\Big) \leq \\
&\Big(\varphi_1(u_K^{n+1}) - \varphi_1(u_{K,K|L}^{n+1})\Big) C_\eta \Big(\pi_1(u_K^{n+1}) - \pi_2(u_L^{n+1})\Big).
\end{aligned}
\tag{2.15}
$$

Then, from (2.9) and (2.15), we get

$$\sum_{n=0}^{M} \delta t \sum_{(K,L)\in\mathcal{T}_\Gamma} \tau_{K,K|L} \Big(\Psi(\hat{\pi}_1(u_K^{n+1})) - \Psi(\hat{\pi}_1(u_{K,K|L}^{n+1}))\Big)^2 \leq C_\eta C_1\,,$$

and in the same way

$$\sum_{n=0}^{M} \delta t \sum_{(K,L)\in\mathcal{T}_\Gamma} \tau_{L,K|L} \Big(\Psi(\hat{\pi}_2(u_{L,K|L}^{n+1})) - \Psi(\hat{\pi}_2(u_L^{n+1}))\Big)^2 \leq C_\eta C_1\,.$$

Thus we get

$$\sum_{n=0}^{M} \delta t \sum_{(K,L)\in\mathcal{T}_\Gamma} \tau_{K|L} (w_K^{n+1} - w_L^{n+1})^2 \leq 2 C_1\, C_\eta.$$

Gathering the above results prove that there exists $C_6 > 0$, only depending on η_j, π_j, Ω_j, $j \in \{1,2\}$ such that

$$|w_{\mathcal{D}}|^2_{1,\mathcal{D}} \leq C_6$$

QED. □

2.6.4. Convergence of the scheme toward the weak problem. Thanks to the previous propositions, we are now able to prove the following theorem which states the convergence of the scheme (2.2)–(2.4) towards a solution to the weak problem introduced in Definition 1.3.

THEOREM 2.15. *Under Assumptions 1.1, let us consider a sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$, of admissible discretizations in the sense of Definition 2.3, such that there exists $\alpha > 0$ with* $\mathrm{regul}(\mathcal{M}_\mathrm{m}) \leq \alpha$ *for all $m \in \mathbb{N}$ and such that* $\mathrm{size}(\mathcal{D}_m) \to 0$ *as $m \to +\infty$. Let $u_{\mathcal{D}_m} = u_m \in \mathcal{X}(\mathcal{D}_m)$ be the solution of the equations (2.2)–(2.4) for $\mathcal{D} = \mathcal{D}_m$. Then there exists a subsequence of $(\mathcal{D}_m, u_m)_{m\in\mathbb{N}}$, again denoted by $(\mathcal{D}_m, u_m)_{m\in\mathbb{N}}$, and a weak solution u of problem (1.5)–(1.9) in the sense of Definition 1.3, such that $u_m \to u$ in $L^p(\Omega \times (0,T))$ for all $p < \infty$.*

Remark 2.16. A proof that the problem (1.5)–(1.9) admits at most one regular solution can be obtained following the method of [5]. A uniqueness result on the solution of the weak problem given in Definition 1.3 implies that the whole sequence of discrete solutions converges.

Proof.

Step 1: Existence of a convergent subsequence of $(\mathcal{D}_m, u_m)_{m\in\mathbb{N}}$.

For any open subset ω_i of Ω_i, $i = 1,2$, Propositions 2.7, 2.11 and 2.12 ensure that the hypotheses of Kolmogorov's theorem are satisfied. We thus get the existence

of a subsequence of $(\varphi_{\mathcal{D}_m,\omega_i})_{m\in\mathbb{N}}$, converging in $L^2(\omega_i\times(0,T))$ to some function $\varphi_{\omega_i}\in L^2(\omega_i\times(0,T))$. Using an increasing sequence of domains $\omega_{i,k}$ which converges towards Ω_i, we can extract, thanks to a diagonal process, a subsequence again denoted by $(\mathcal{D}_m,u_m)_{m\in\mathbb{N}}$ such that $(\varphi_{\mathcal{D}_m,\omega_{i,m}})_{m\in\mathbb{N}}$ converges in $L^2(\omega_{i,k}\times(0,T))$ for all $k\in\mathbb{N}$, to some bounded function $\tilde{\varphi}_i\in L^2(\omega_{i,k}\times(0,T))$ for all $k\in\mathbb{N}$. We then obtain that $(\varphi_i(u_m))_{m\in\mathbb{N}}$ converges in $L^2(\Omega_i\times(0,T))$ to $\tilde{\varphi}_i$. Since φ_i is continuous and strictly increasing, this implies that, up to a subsequence, $(u_m)_{m\in\mathbb{N}}$ converges towards a function $u_i\in L^2(\Omega_i\times(0,T))\bigcap L^\infty(\Omega_i\times(0,T))$ for all $i\in\{1,2\}$.
To prove that $\varphi_i(u_i)\in L^2(0,T;H^1(\Omega_i))$ for all $i\in\{1,2\}$, it is sufficient to show that $\frac{\partial\varphi_i(u_i)}{\partial x}\in L^2(\Omega_i\times(0,T))$. Let $m\in\{0\dots M\}$, $\psi_i\in C_c^\infty(\Omega_i\times(0,T))$ and $\epsilon>0$ be such that $\mathrm{supp}(\psi_i)=\{(x,t)\in\Omega_i\times(0,T)\ /\ \mathrm{dist}(\mathrm{x},\mathbb{R}^\mathrm{d}\setminus\Omega_\mathrm{i})\leq\epsilon\}$. Using the Cauchy-Schwarz inequality and the Lemma 2.10 we have, for all $|\xi|\leq\epsilon$,

$$\int_{\Omega_{i,\xi}\times(0,T)}\Big(\varphi_i(u_m(x+\xi,t))-\varphi_i(u_m(x,t))\Big)\psi_i(x,t)dxdt\leq$$
$$\Big(|\xi|(|\xi|+2\mathrm{size}(\mathcal{M}_\mathrm{m}))\mathrm{C}_2\Big)^{\frac{1}{2}}\|\psi_\mathrm{i}\|_{\mathrm{L}^2(\Omega_\mathrm{i}\times(0,\mathrm{T}))}.$$

Passing to the limit and after a change of variable we obtain

$$\int_{\Omega_{i,\xi}\times(0,T)}\Big(\psi_i(x-\xi,t)-\psi_i(x,t)\Big)\varphi_i(u_i(x,t))dxdt\leq$$
$$|\xi|(C_2)^{\frac{1}{2}}||\psi_i||_{L^2(\Omega_i\times(0,T))}. \tag{2.16}$$

Now if we denote by $\{e_i, i=1\dots d\}$ the canonical basis of $\mathbb{R}^d$ and if we take $\xi=\lambda e_i,\ i\in\{1\dots d\}$ with $|\lambda|<\epsilon$ in (2.16), we then have as $\epsilon\to0$

$$-\int_{\Omega_{i,\xi}\times(0,T)}\frac{\partial\psi_i(x,t)}{\partial x_i}\varphi_i(u_i(x,t))dxdt\leq(C_2)^{\frac{1}{2}}\|\psi_i\|_{L^2(\Omega_i\times(0,T))},$$
$$\forall\psi_i\in C_c^\infty(\Omega_i\times(0,T)),$$

which implies that $\frac{\partial\varphi_i(u_i)}{\partial x}\in L^2(\Omega_i\times(0,T))$.

Step 2: u is a weak solution to the problem (1.5)–(1.9).

Let us consider $\tilde{C}_{test}=\{h\in C^2(\overline{\Omega}\times[0,T])\ /\ h(.,T)=0\}$ which is dense in C_{test}. Let $\psi\in\tilde{C}_{test}$ and, for $m\in\mathbb{N}$, let u_m be given by the equations (2.2)–(2.4) for $\mathcal{D}=\mathcal{D}_m$. For all $n\in\{0\dots M\}$ and for all $K\in\mathcal{T}$, we multiply the equation (2.3) by $\psi_K^n=\psi(x_K,n\delta t)$, and we sum these equalities over the volume control set and $n=0,\dots,M$. We get $\sum_{i=1}^2(E_{i,1,m}+E_{i,2,m})+E_{1|2,m}=0$, with

$$E_{i,1,m}=\sum_{n=0}^{M}\sum_{K\in\mathcal{T}_i}m(K)\phi_i(u_K^{n+1}-u_K^n)\psi_K^n,$$
$$E_{i,2,m}=-\sum_{n=0}^{M}\delta t\sum_{K\in\mathcal{T}_i}\sum_{L\in N(K)}\tau_{K|L}\Big(\varphi_i(u_L^{n+1})-\varphi_i(u_K^{n+1})\Big)\psi_K^n,$$
$$E_{1|2,m}=\sum_{n=0}^{M}\delta t\sum_{(K,L)\in\mathcal{T}_\Gamma}\tau_{K,K|L}\left(\varphi_1(u_K^{n+1})-\varphi_1(u_{K,K|L}^{n+1})\right)\Big(\psi_K^n-\psi_L^n\Big).$$

Following some classical proofs (see [11]), we get that

$$\lim_{m\to+\infty} E_{i,1,m} = -\int_0^T \int_{\Omega_i} \phi_i u_i(x,t)\psi_t(x,t)dxdt - \int_{\Omega_i} \phi_i u_{\mathrm{ini}}(x)\psi(x,0)dx.$$

Convergence of $E_{i,2,m}$:

Gathering the terms by edges in $E_{i,2,m}$ leads to

$$E_{i,2,m} = \sum_{n=0}^{M} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{int,i}} \tau_{K|L}\Big(\varphi_i(u_K^{n+1}) - \varphi_i(u_L^{n+1})\Big)\Big(\psi_K^n - \psi_L^n\Big).$$

We apply the method presented, for example in [10] (which is a discrete version of a strong-weak convergence), to conclude that

$$\lim_{m\to+\infty} E_{i,2,m} = \int_0^T \int_{\Omega_i} \nabla\varphi_i(u_i)(x,t).\nabla\psi(x,t)\,dx\,dt. \tag{2.17}$$

Convergence of $E_{1|2,m}$:

We have

$$E_{1|2,m}^2 \leq \left(\sum_{n=0}^{M} \delta t \sum_{(K,L)\in\mathcal{T}_\Gamma} \tau_{K,K|L}\Big(\varphi_1(u_K^{n+1}) - \varphi_1(u_{K,K|L}^{n+1})\Big)^2\right) \times \left(\sum_{n=0}^{M} \delta t \sum_{(K,L)\in\mathcal{T}_\Gamma} m(K|L)\frac{(\psi_K^n - \psi_L^n)^2}{d_{K,K|L}}\right).$$

But we notice that, thanks to the regularity of the function ψ, there exists $C_\psi > 0$ such that $|\psi_K^n - \psi_L^n| \leq C_\psi d_{K|L}$, which implies with (2.1)

$$\sum_{n=0}^{M} \delta t \sum_{(K,L)\in\mathcal{T}_\Gamma} m(K|L)\frac{(\psi_K^n - \psi_L^n)^2}{d_{K,K|L}} \leq 4Tm(\Gamma)C_\psi^2\alpha\mathrm{size}(\mathcal{M}).$$

Thus, using (2.9) and (2.15), we get

$$\sum_{n=0}^{M} \delta t \sum_{(K,L)\in\mathcal{T}_\Gamma} \tau_{K,K|L}\Big(\varphi_i(u_K^{n+1}) - \varphi_i(u_{K,K|L}^{n+1})\Big)^2 \leq C_\eta C_1\,.$$

Gathering the above results produces

$$\lim_{m\to+\infty} E_{1|2,m} = 0.$$

Step 3: Let us prove that $w \in L^2(0,T;H^1(\Omega))$.

Following the proofs of Lemma 2.10 and of $\varphi(u_i) \in L^2(0,T;H^1(\Omega_i))$ (see Step 1), we obtain that $w \in L^2(0,T;H^1(\Omega))$ using inequality (2.14). ∎

As an immediate consequence of Theorem 2.15 we get

COROLLARY 2.17. *Under Assumptions 1.1, Problem (1.5)–(1.9) admits at least one weak solution in the sense of Definition 1.3.*

As an illustration of the previous results, we now give numerical results in the following section.

3. Numerical results. Let us consider a domain Ω such that $\Omega_1 = (0,1)$ and $\Omega_2 = (1,2)$. The mobilities are given by

$$\eta_o(u) = \begin{cases} u & \text{if } 0 \le u \le 1, \\ 0 & \text{if } u < 0, \\ 1 & \text{otherwise,} \end{cases} \qquad \eta_w(u) = \begin{cases} 1-u & \text{if } 0 \le u \le 1, \\ 1 & \text{if } u < 0, \\ 0 & \text{otherwise} \end{cases}$$

and the capillary pressure is given by

$$\pi_1(u) = \begin{cases} 5u^2 & \text{if } 0 \le u \le 1, \\ 0 & \text{if } u < 0, \\ 5 & \text{otherwise,} \end{cases} \qquad \pi_2(u) = \begin{cases} 5u^2+1 & \text{if } 0 \le u \le 1, \\ 1 & \text{if } u < 0, \\ 6 & \text{otherwise.} \end{cases}$$

In that case, $u_1^\star = \frac{1}{\sqrt{5}}$, $u_2^\star = \frac{2}{\sqrt{5}}$. For the initial condition we take

$$u_{\text{ini}}(x) = \begin{cases} 0.9 & \text{if} \qquad x < 0.9, \\ 0 & \text{otherwise.} \end{cases}$$

To discretize the domains Ω_i, we use a regular mesh such that $\mathrm{dx} = \mathrm{size}(\mathcal{M}) = 10^{-2}$ for all $i \in \{1,2\}$ and we use a constant time step $\delta t = \frac{1}{6}.10^{-3}$. Figures 3.1 represent functions $u(.,t)$, $\pi(.,u(.,t))$, $\varphi(.,u(.,t))$ for $t = 0.007$ and $t = 0.05$. In the first case oil is trapped under the interface Γ located in $x = 1$ and the capillary pressure is discontinuous whereas in the second case oil can flow through Γ and the continuity of the capillary pressure is ensured. Figure 3.2 represents the evolution of the flux and of the saturations on the interface Γ according to the time variable. We have also done tests with the initial condition

$$u_{\text{ini}}(x) = \begin{cases} 0.9 & \text{if} \qquad x > 1.2, \\ 0 & \text{otherwise} \end{cases}$$

where oil already lies in the capillary barrier. Figures 3.3, 3.4 show the results we obtained. We notice that, although the capillary pressure is discontinuous, oil can flow through Γ from Ω_2 to Ω_1 while satisfying the conditions (2.4) since, for all $t \in [0, 0.05]$, $u_2(t) = 0$.

4. Concluding remarks. In this paper we have established a convergence property for the scheme (2.2)–(2.4) towards a weak solution of the problem (1.5)–(1.9) in the sense of Definition 1.3. It remains to prove the uniqueness of such a weak solution. Further works will be done with taking a total flux and the gravity gradient into account (see [8]).

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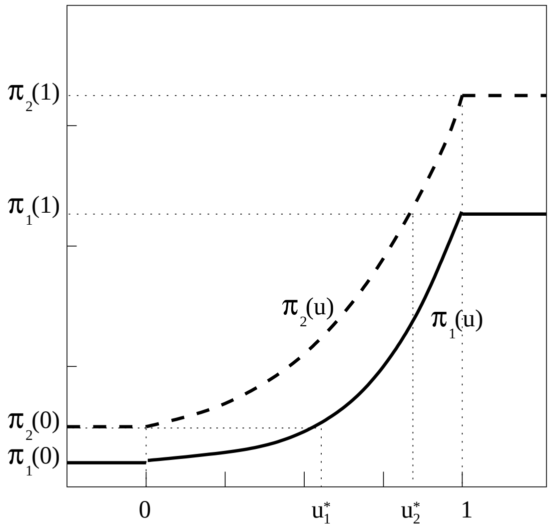
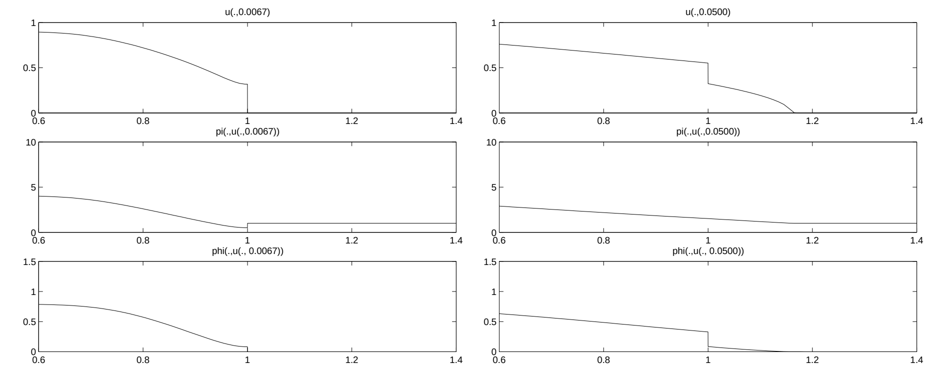


Fig. 1.1. *Functions* π_i, $i = 1, 2$



(a) (b)

Fig. 3.1. $u(.,t)$, $\pi(.,u(.,t))$, $\varphi(.,u(.,t))$ *for* $t = 0.007$ *(a) and* $t = 0.05$ *(b).*

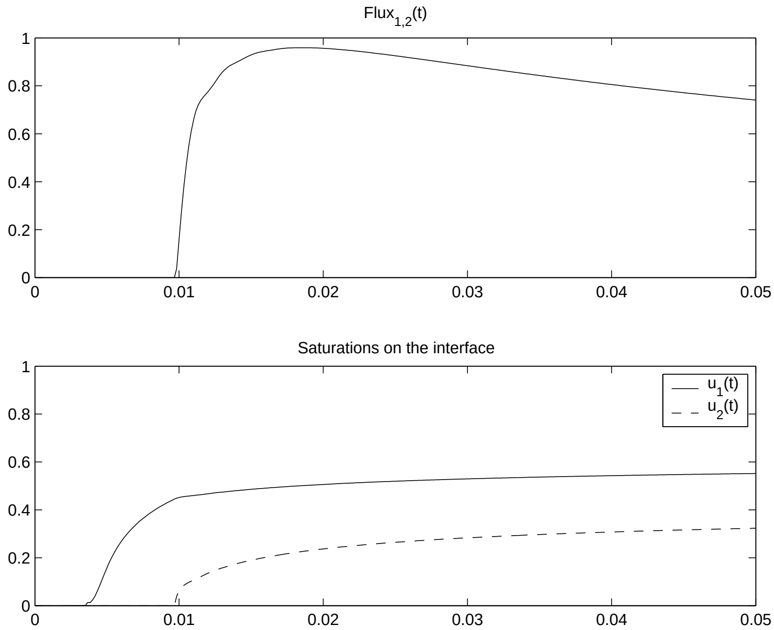
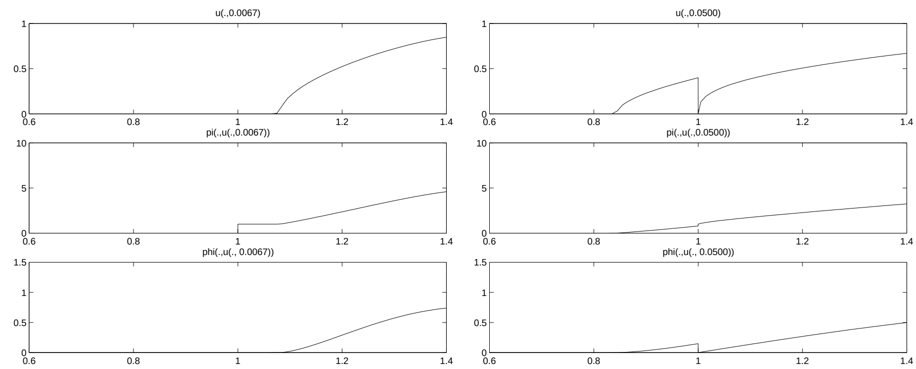


FIG. 3.2. *Evolution of the flux and of the saturations on the interface.*



(a) (b)

FIG. 3.3. $u(.,t)$, $\pi(.,u(.,t))$, $\varphi(.,u(.,t))$ *for* $t = 0.007$ *(a) and* $t = 0.05$ *(b).*

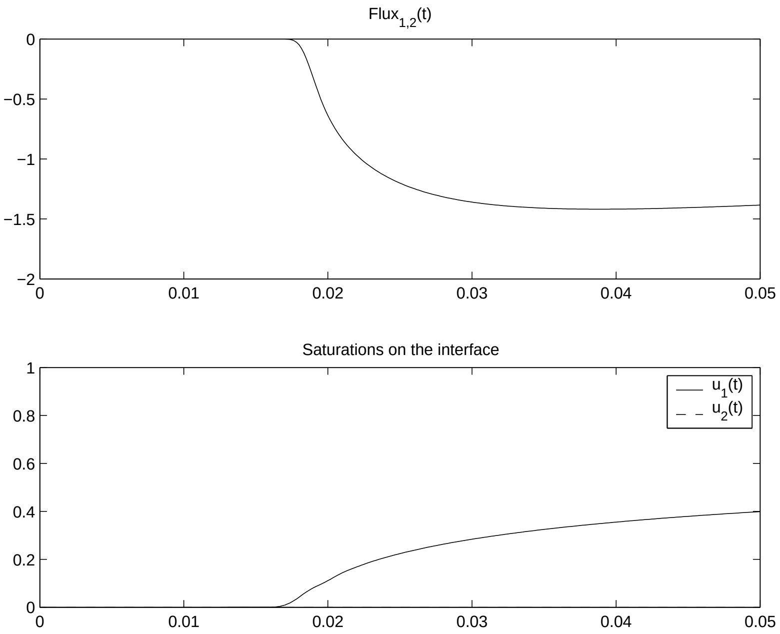


Fig. 3.4. *Evolution of the flux and of the saturations on the interface.*