A mathematical and numerical study of an industrial scheme for two-phase flows in porous media under gravity

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The continuous model

The coupled system

\[ \begin{cases} 
\phi \frac{\partial s}{\partial t} + \text{div} \left( \Upsilon(\phi) \eta_o(s)(\rho_o \vec{g} - \vec{\nabla}p) \right) = 0 \text{ on } \Omega \times (0, T) \\
-\phi \frac{\partial s}{\partial t} + \text{div} \left( \Upsilon(\phi) \eta_w(s)(\rho_w \vec{g} - \vec{\nabla}p) \right) = 0 \text{ on } \Omega \times (0, T) 
\end{cases} \]

Boundary conditions

\[ \forall \alpha \in \{o, w\}, \ \Upsilon(\phi) \left( \eta_\alpha(s)(\rho_\alpha \vec{g} - \vec{\nabla}p) \right) \cdot \vec{n} = 0 \text{ on } \partial \Omega \times (0, T) \]

Initial condition

\[ s(x, 0) = s_{ini}(x) \]
The upstream weighting scheme

The upstream weighting scheme (explicit case)

For all $n \in \{0 \ldots M\}$ and $K \in T$ we have

\[
\begin{cases}
   m(K)\phi_K \frac{s_{K}^{n+1} - s_{K}^{n}}{\delta t} + \sum_{L \in N(K)} \gamma_{K|L}(\eta_{o})_{K|L}^{n+1}(\rho_{o}g\delta z_{K,L} - \delta p_{K,L}^{n+1}) = 0, \\
   -m(K)\phi_K \frac{s_{K}^{n+1} - s_{K}^{n}}{\delta t} + \sum_{L \in N(K)} \gamma_{K|L}(\eta_{w})_{K|L}^{n+1}(\rho_{w}g\delta z_{K,L} - \delta p_{K,L}^{n+1}) = 0,
\end{cases}
\]

where

\[
(\eta_{\alpha})_{K|L}^{n+1} = \begin{cases}
    (\eta_{\alpha})_{K}^{n+1} = \eta_{\alpha}(S_{K}^{n}) & \text{if } \rho_{\alpha}g\delta z_{K,L} - \delta p_{K,L}^{n+1} \geq 0, \\
    (\eta_{\alpha})_{L}^{n+1} = \eta_{\alpha}(S_{L}^{n}) & \text{otherwise}.
\end{cases}
\]
The upstream weighting scheme

The decoupled form

Summing both equations we get

\[
\forall n \in \{0 \ldots M\}, \forall K \in \mathcal{T}, \sum_{L \in \mathcal{N}(K)} Q_{K,L}^{n+1} = 0
\]

with

\[
Q_{K,L}^{n+1} = \Upsilon_{K|L} \left( (\eta_o)^{n+1}_{K|L} \rho_o g + (\eta_w)^{n+1}_{K|L} \rho_w g \right) \delta z_{K,L} - \left( (\eta_o)^{n+1}_{K|L} + (\eta_w)^{n+1}_{K|L} \right) \delta p_{K,L}^{n+1}.
\]

Expressing \( \delta p_{K,L}^{n+1} \) in terms of \( Q_{K,L}^{n+1} \), plugging this expression into the "oil equation", and setting \( G_{K,L} = \Upsilon_{K|L} (\rho_o - \rho_w)g \delta z_{K,L} \) we obtain

\[
m(K) \phi_K \frac{s_{K}^{n+1} - s_{K}^{n}}{\delta t} + \sum_{L \in \mathcal{N}(K)} \frac{(\eta_o)^{n+1}_{K|L} \left( Q_{K,L}^{n+1} + (\eta_w)^{n+1}_{K|L} G_{K,L} \right)}{\left( (\eta_o)^{n+1}_{K|L} + (\eta_w)^{n+1}_{K|L} \right)} = 0.
\]

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The discrete oil flux

We set

\[ F(s^n_K, s^n_L, Q^{n+1}_{K,L}, G_{K,L}) = \frac{(\eta_o)^{n+1}_{K|L} \left( Q^{n+1}_{K,L} + (\eta_w)^{n+1}_{K|L} G_{K,L} \right)}{(\eta_o)^{n+1}_{K|L} + (\eta_w)^{n+1}_{K|L}}. \]

Properties of \( F(., ., Q, G) \):

- The upwind saturations can be determined only with the values of \( Q \) and \( G \).

- The oil flux is monotonous: \( F(., ., Q, G) \) is nondecreasing with respect to its first argument and nonincreasing with respect to its second argument.
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\textbf{\textit{L}}^\infty \textbf{ stability of the saturation calculation}

Using the monotonicity of the flux we can show that, for all \(n \in \{0 \ldots M\}\) and \(K \in T\), we have

\[0 \leq s^n_K \leq 1\]

in the implicit case as well as in the explicit case under the CFL condition

\[
\delta t \leq \inf_{K \in T} \left( \frac{m(K)}{\sum_{L \in N(K)} C_\eta(|Q_{K,L}^{n+1}| + |G_{K,L}|)} \right).
\]

\[\text{Austin, March 2003}\]
**$L^2$ pressure estimate**

We first prove a discrete $H^1$-seminorm on the pressure

$$|p_{\mathcal{M}}^{n+1}|_{1,\mathcal{M}} = \sum_{K|L\in \mathcal{E}_{int}} \tau_K|L(\delta_{p_K,L}^{n+1})^2 \leq C.$$  

From the Poincaré-Wirtinger inequality we deduce that

$$\|p_{\mathcal{M}}^{n+1}\|_{L^2(\Omega)} = \sum_{K\in \mathcal{T}} m(K)(p_K^{n+1})^2 \leq C.$$  

**Remark:** The $H^1$-seminorm on the pressure ensures that there exists a time step $\delta t > 0$ satisfying the previous CFL condition.
Existence of solutions to the discrete systems

The explicit case The system is nonlinear in pressure because of the
saturation upwinding. The $L^2$ pressure estimate and a topological
degree argument ensure the existence of a couple of solutions
$$(s_{K}^{n+1}, p_{K}^{n+1})_{K \in \mathcal{T}} \text{ for all } n \in \{0 \ldots M\}.$$ 

The implicit case The same arguments used with the saturation and
the pressure estimates give the result.
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Numerical convergence: Error in saturation

Error in saturation (L1 norm)

Linear regression: slope = 0.8616

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Numerical convergence: Error in pressure

Error in pressure (L2 norm)

Linear regression: slope = 2.0395

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A 2D-case
A 2D-case
Numerical results

A 2D-case

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Numerical results

A 2D-case

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Concluding remarks

Convergence (1)

• Numerically the scheme is convergent.

• The mathematical proof is still a challenge. Two difficulties:

  1. From the saturation estimate we deduce that, up to a subsequence,

     \( s_M \rightarrow \overline{s}, \)

     \( k^\alpha_r, M \rightarrow \overline{k^\alpha_r} \)

     in the weak-\(\star\) sense. But, because of the nonlinearities,

     \( \overline{k^\alpha_r} \neq k^\alpha_r(\overline{s}). \)
Concluding remarks

Convergence (2)

2. From the discrete $H^1$-seminorm, we deduce that, up to a subsequence and for all $t \in (0, T)$,

\[
\begin{align*}
\mathcal{R}M(. , t) & \rightharpoonup \overline{p}_t, \\
\nabla \mathcal{R}M(. , t) & \rightharpoonup \nabla \overline{p}_t
\end{align*}
\]

weakly in $L^2(\Omega)$. Thus both mobilities and pressure gradients converge in weak senses but we can not conclude about their product.

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