A mathematical and numerical study of an industrial scheme for two-phase flows in porous media under gravity

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The continuous model

The coupled system

$$\left\{ \begin{array}{l} \phi\dfrac{\partial s}{\partial t} + \mathrm{div}\Big(\Upsilon(\phi)\ \eta_o(s)(\rho_o \overrightarrow{g} - \overrightarrow{\nabla} p)\Big) = 0 \text{ on } \Omega \times (0,T) \\ -\phi\dfrac{\partial s}{\partial t} + \mathrm{div}\Big(\Upsilon(\phi)\ \eta_w(s)(\rho_w \overrightarrow{g} - \overrightarrow{\nabla} p)\Big) = 0 \text{ on } \Omega \times (0,T) \end{array} \right.$$

Boundary conditions

$$\forall \alpha \in \{o, w\},\ \Upsilon(\phi) \left(\eta_\alpha(s)(\rho_\alpha \overrightarrow{g} - \overrightarrow{\nabla} p)\right).\overrightarrow{n} = 0 \text{ on } \partial\Omega \times (0,T)$$

Initial condition

$$s(x,0) = s_{ini}(x)$$

The upstream weighting scheme (explicit case)

For all $n \in \{0 \ldots M\}$ and $K \in \mathcal{T}$ we have

$$\begin{cases} m(K)\phi_K \dfrac{s_K^{n+1} - s_K^n}{\delta t} + \displaystyle\sum_{L \in N(K)} \Upsilon_{K|L}(\eta_o)_{K|L}^{n+1}(\rho_o g \delta z_{K,L} - \delta p_{K,L}^{n+1}) = 0, \\ -m(K)\phi_K \dfrac{s_K^{n+1} - s_K^n}{\delta t} + \displaystyle\sum_{L \in N(K)} \Upsilon_{K|L}(\eta_w)_{K|L}^{n+1}(\rho_w g \delta z_{K,L} - \delta p_{K,L}^{n+1}) = 0, \end{cases}$$

where

$$(\eta_\alpha)_{K|L}^{n+1} = \begin{cases} (\eta_\alpha)_K{}^{n+1} = \eta_\alpha(S_K{}^n) & \text{if } \rho_\alpha g \delta z_{K,L} - \delta p_{K,L}^{n+1} \geq 0, \\ (\eta_\alpha)_L{}^{n+1} = \eta_\alpha(S_L{}^n) & \text{otherwise.} \end{cases}$$

The decoupled form

Summing both equations we get

$$\forall n \in \{0 \dots M\}, \forall K \in \mathcal{T}, \quad \sum_{L \in N(K)} Q_{K,L}^{n+1} = 0$$

with

$$Q_{K,L}^{n+1} = \Upsilon_{K|L} \left(\left((\eta_o)_{K|L}^{n+1} \rho_o g + (\eta_w)_{K|L}^{n+1} \rho_w g \right) \delta z_{K,L} - \left((\eta_o)_{K|L}^{n+1} + (\eta_w)_{K|L}^{n+1} \right) \delta p_{K,L}^{n+1} \right).$$

Expressing $\delta p_{K,L}^{n+1}$ in terms of $Q_{K,L}^{n+1}$, plugging this expression into the "oil equation", and setting $G_{K,L} = \Upsilon_{K|L}(\rho_o - \rho_w) g \delta z_{K,L}$ we obtain

$$m(K)\phi_K \frac{s_K^{n+1} - s_K^n}{\delta t} + \sum_{L \in N(K)} \frac{(\eta_o)_{K|L}^{n+1} \left(Q_{K,L}^{n+1} + (\eta_w)_{K|L}^{n+1} G_{K,L} \right)}{(\eta_o)_{K|L}^{n+1} + (\eta_w)_{K|L}^{n+1}} = 0.$$

The discrete oil flux

We set

$$F(s_K^n, s_L^n, Q_{K,L}^{n+1}, G_{K,L}) = \frac{(\eta_o)_{K|L}^{n+1}\left(Q_{K,L}^{n+1} + (\eta_w)_{K|L}^{n+1} G_{K,L}\right)}{(\eta_o)_{K|L}^{n+1} + (\eta_w)_{K|L}^{n+1}}.$$

Properties of $F(.,.,Q,G)$:

- The upwind saturations can be determined only with the values of Q and G.
- The oil flux is monotonous : $F(.,.,Q,G)$ is nondecreasing with respect to its first argument and nonincreasing with respect to its second argument.

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L^∞ stability of the saturation calculation

Using the monotonicity of the flux we can show that, for all $n \in \{0 \ldots M\}$ and $K \in \mathcal{T}$, we have

$$\boxed{0 \leq s_K^n \leq 1}$$

in the implicit case as well as in the explicit case under the CFL condition

$$\delta t \leq \inf_{K \in \mathcal{T}} \left(\frac{m(K)}{\displaystyle\sum_{L \in N(K)} C_\eta (|Q_{K,L}^{n+1}| + |G_{K,L}|)} \right).$$

L^2 pressure estimate

We first prove a discrete H^1-seminorm on the pressure

$$|p_{\mathcal{M}}^{n+1}|_{1,\mathcal{M}} = \sum_{K|L \in \mathcal{E}_{int}} \tau_{K|L} (\delta p_{K,L}^{n+1})^2 \leq C.$$

From the Poincaré-Wirtinger inequality we deduce that

$$\|p_{\mathcal{M}}^{n+1}\|_{L^2(\Omega)} = \sum_{K \in \mathcal{T}} m(K)(p_K^{n+1})^2 \leq C.$$

Remark : The H^1-seminorm on the pressure ensures that there exists a time step $\delta t > 0$ satisfying the previous CFL condition.

Existence of solutions to the discrete systems

The explicit case The system is nonlinear in pressure because of the saturation upwinding. The L^2 pressure estimate and a topological degree argument ensure the existence of a couple of solutions $(s_K^{n+1}, p_K^{n+1})_{K\in\mathcal{T}}$ for all $n \in \{0 \dots M\}$.

The implicit case The same arguments used with the saturation and the pressure estimates give the result.

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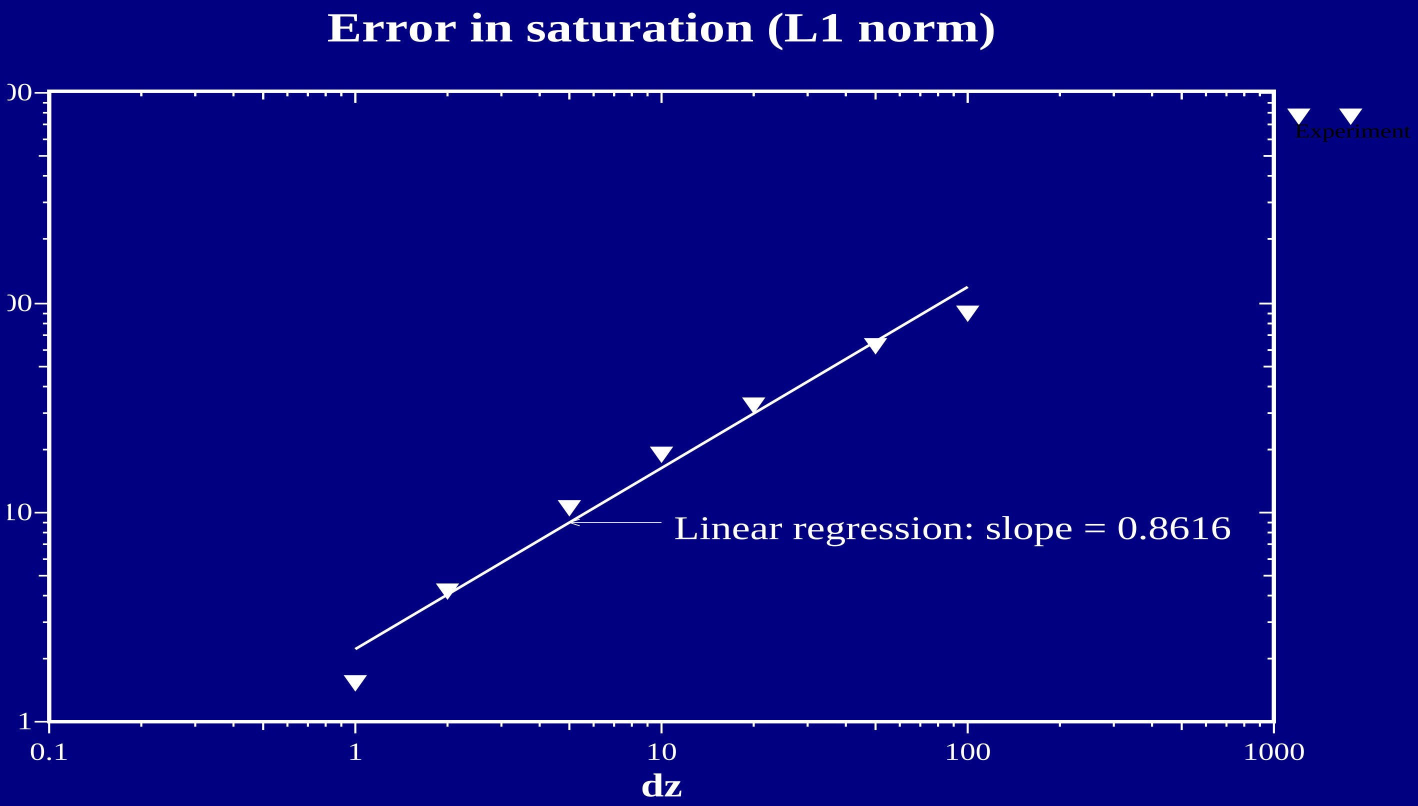
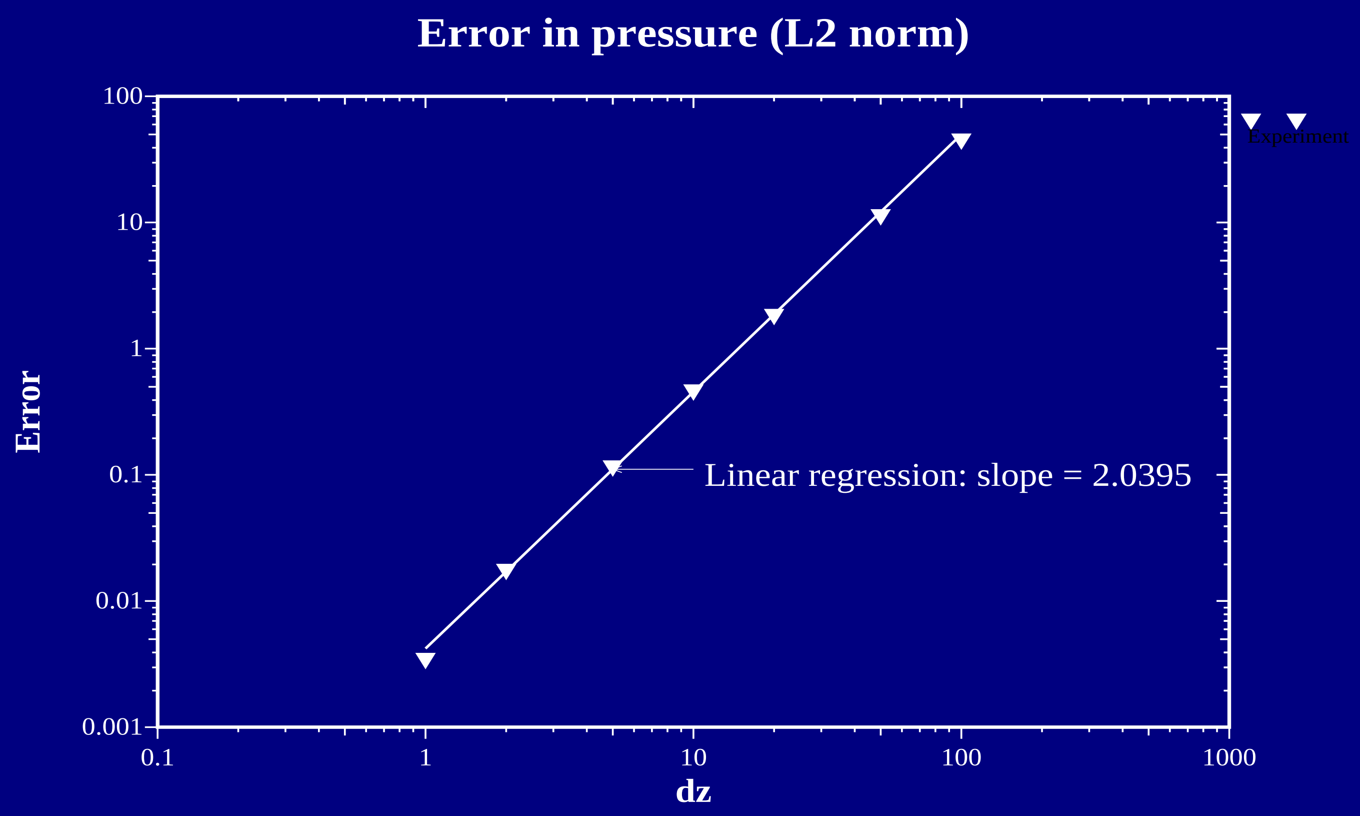
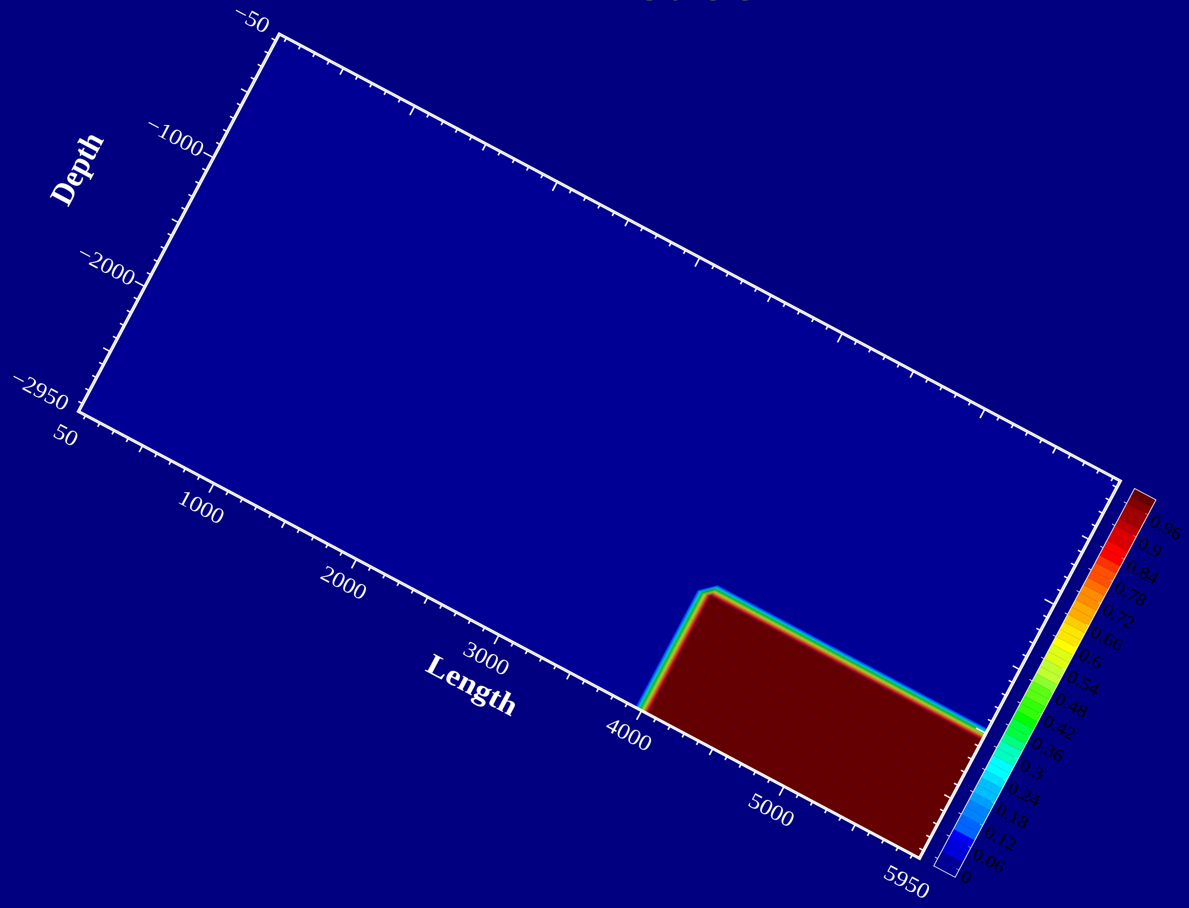
Numerical convergence : Error in saturation



Numerical convergence : Error in pressure



A 2D-case



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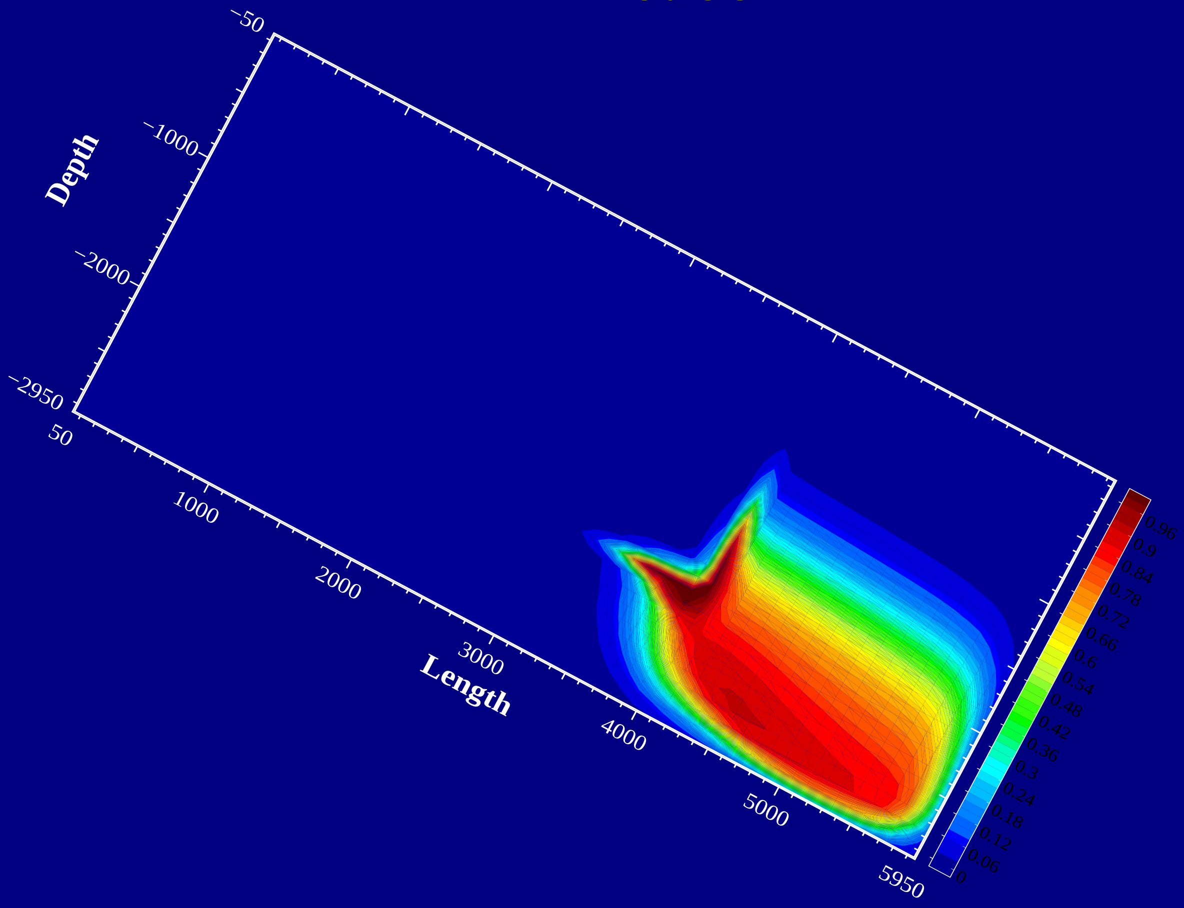
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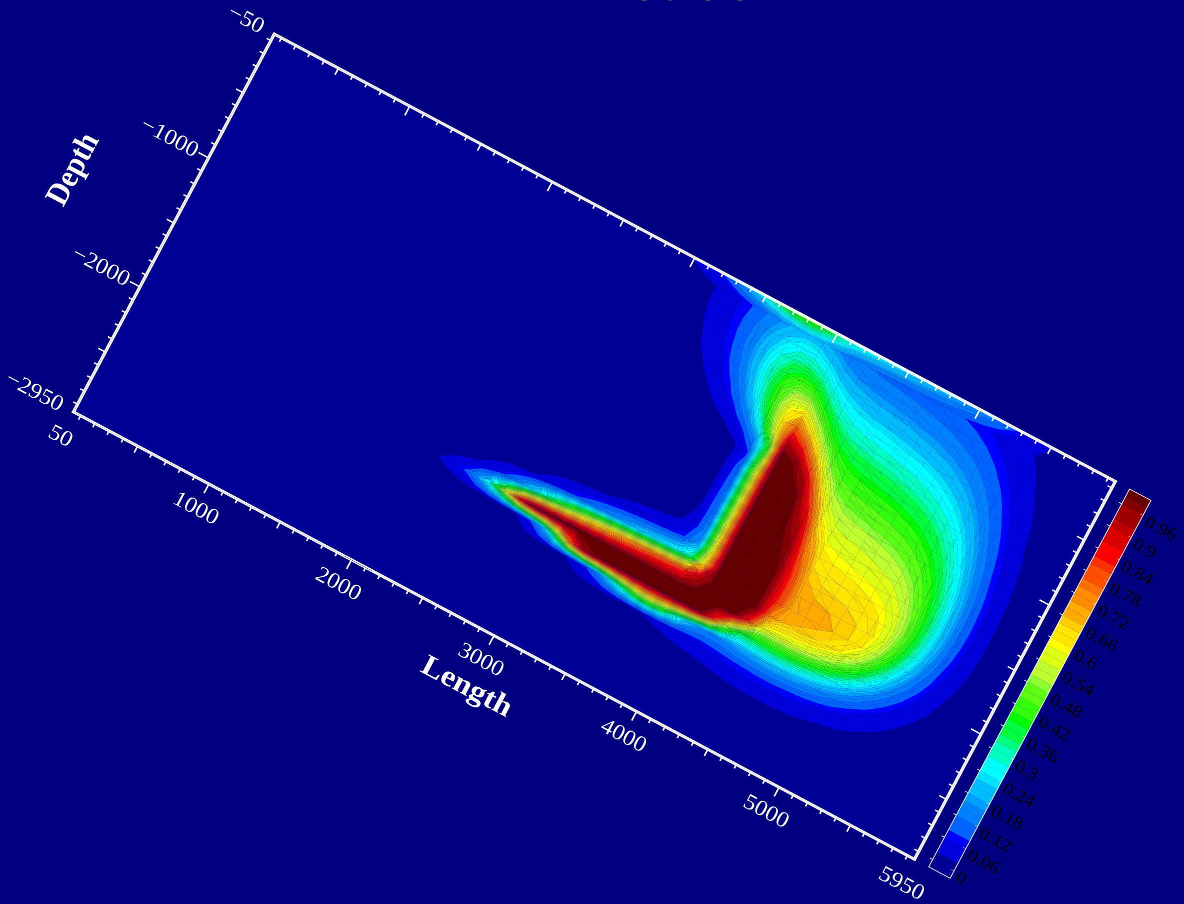
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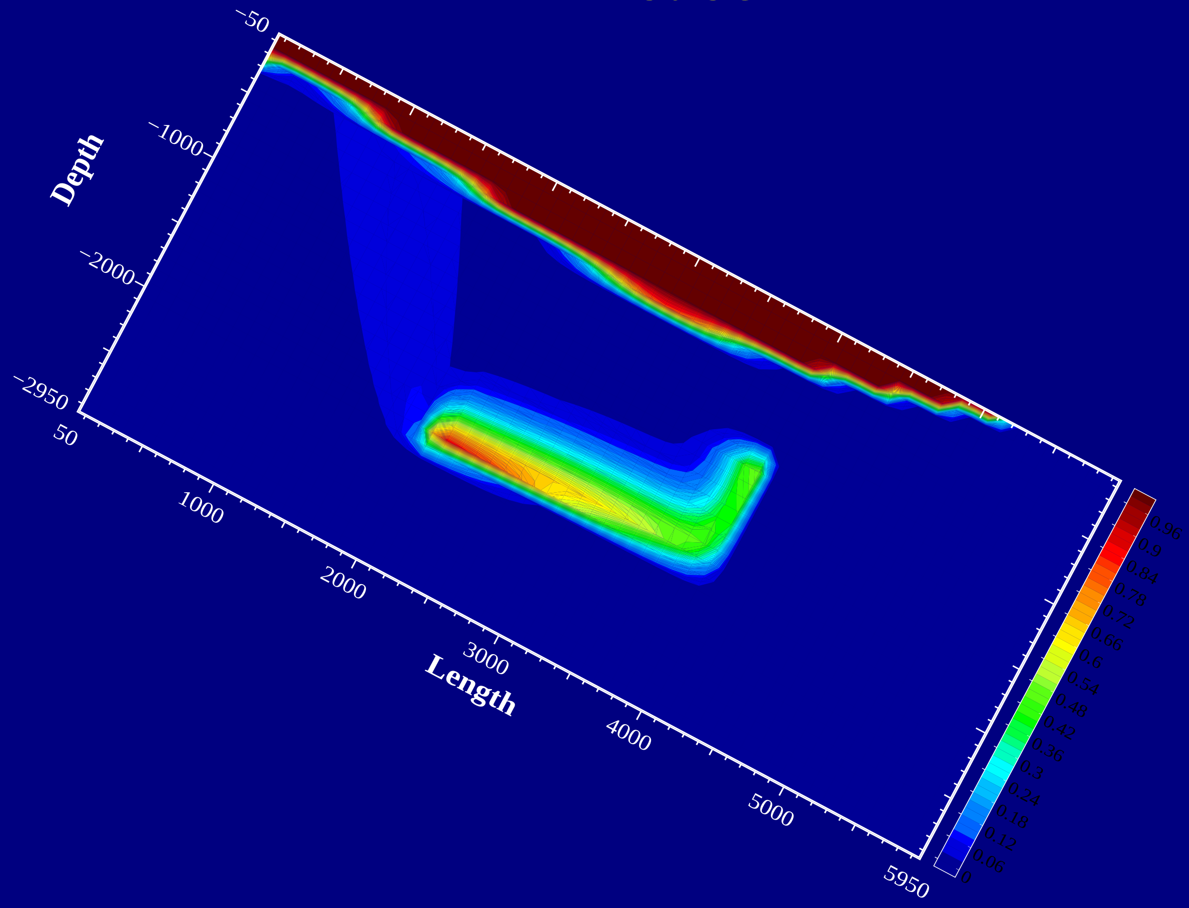
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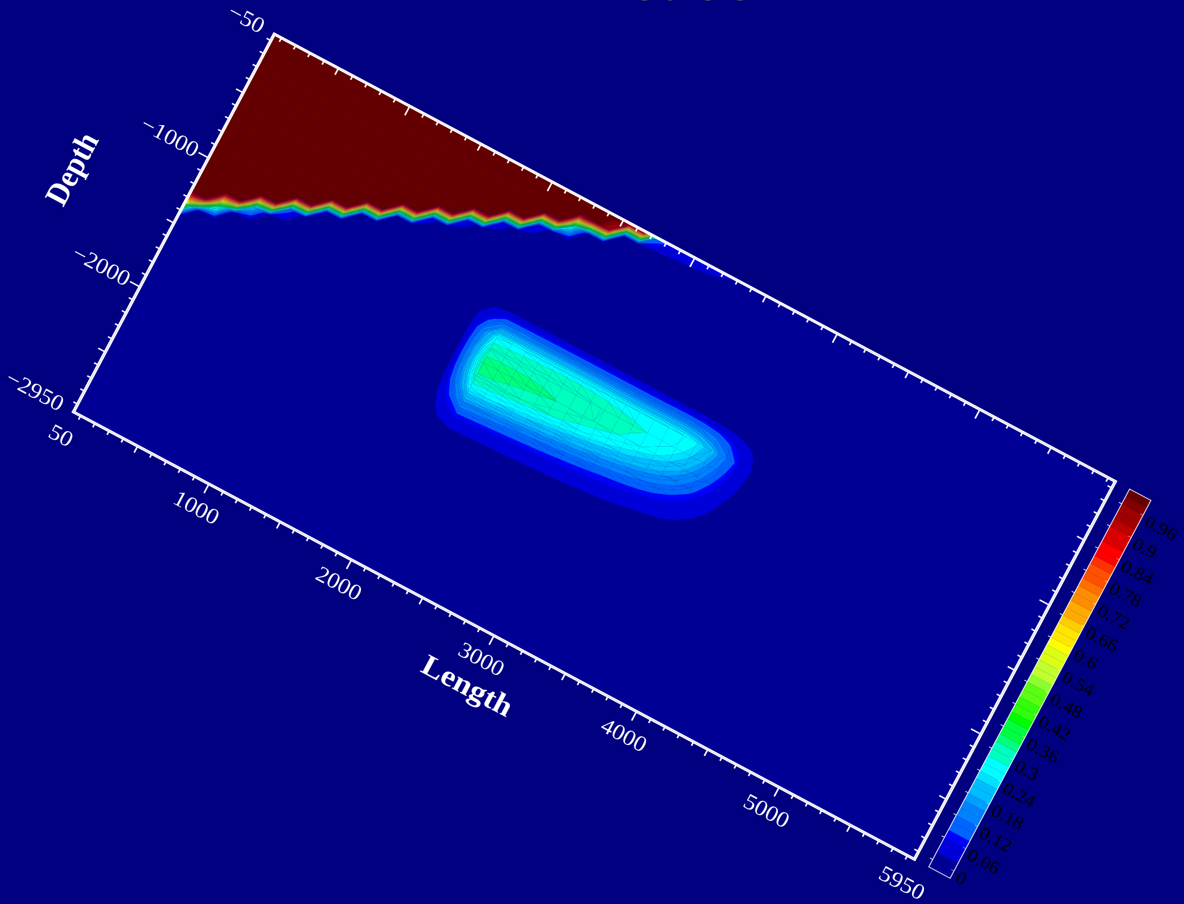
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Convergence (1)

- Numerically the scheme is convergent.
- The mathematical proof is still a challenge. Two difficulties :

1. From the saturation estimate we deduce that, up to a subsequence,

$$s_{\mathcal{M}} \longrightarrow \overline{s}, \qquad kr_{\alpha,\mathcal{M}} \longrightarrow \overline{kr}_{\alpha}$$

in the weak-$\star$ sense. But, because of the nonlinearities,

$$\overline{kr}_{\alpha} \neq kr(\overline{s}).$$

Convergence (2)

2. From the discrete H^1-seminorm, we deduce that, up to a subsequence and for all $t \in (0, T)$,

$$
\boxed{
\begin{array}{l}
p_{\mathcal{M}}(.,t) \rightarrow \overline{p_t}, \\
\overrightarrow{\nabla} p_{\mathcal{M}}(.,t) \rightarrow \overrightarrow{\nabla} \overline{p_t}
\end{array}
}
$$

weakly in $L^2(\Omega)$. Thus both mobilities and pressure gradients converge in weak senses but we can not conclude about their product.