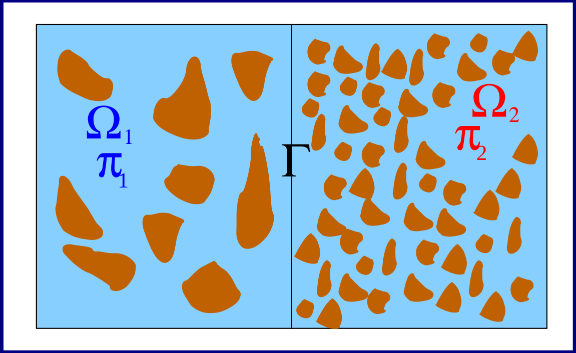
Full Screen

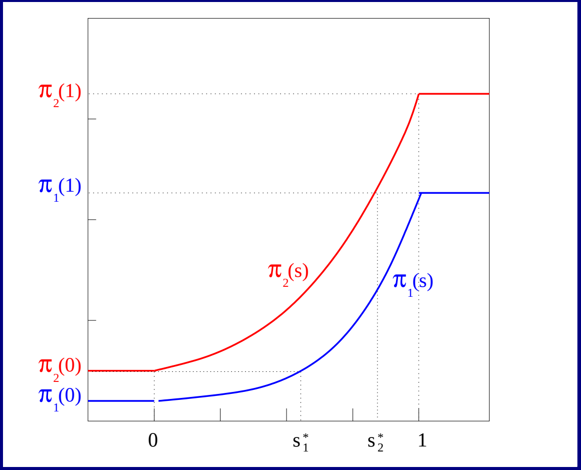
Numerical approximation of a two-phase flow problem in a porous medium with discontinuous capillary forces

G. Enchéry, R. Eymard, A. Michel

1. Capillary trapping

Space discontinuity of the capillary forces





Full Screen

Full Screen

Darcy's equations

$$\begin{cases} \phi \dfrac{\partial s}{\partial t} - \operatorname{div}\Big(\eta_o(.,s)(\overrightarrow{\nabla} p_o - \rho_o \overrightarrow{g})\Big) = 0 \\ -\phi \dfrac{\partial s}{\partial t} - \operatorname{div}\Big(\eta_w(.,s)(\overrightarrow{\nabla} p_w - \rho_w \overrightarrow{g})\Big) = 0 \\ p_o - p_w = \pi(.,s) \end{cases}$$

Physical conditions on the interface

- Conservation of the mass

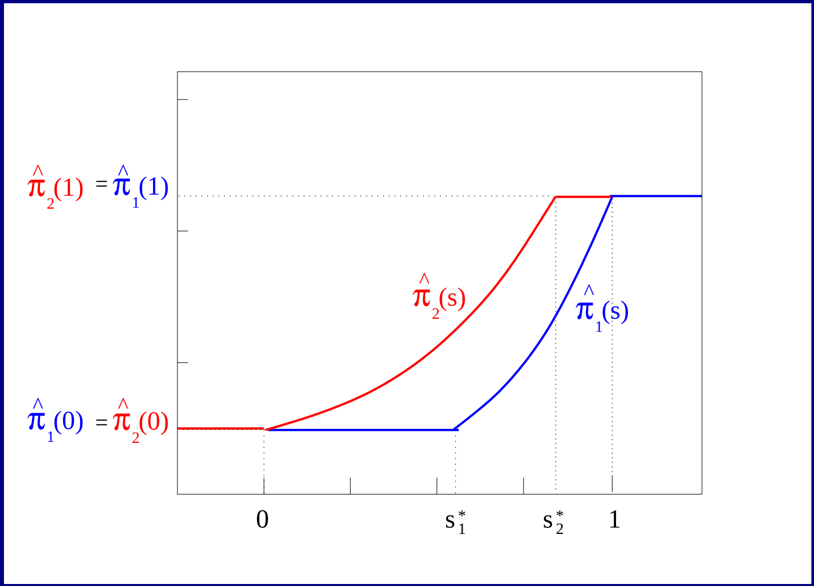
$$\eta_{\beta,1}(s_{1,\Gamma})\Big((\overrightarrow{\nabla} p)_{\beta,1,\Gamma} - \rho_\beta \overrightarrow{g}\Big).\overrightarrow{n}_{1,\Gamma} = -\eta_{\beta,2}(s_{2,\Gamma})\Big((\overrightarrow{\nabla} p)_{\beta,2,\Gamma} - \rho_\beta \overrightarrow{g}\Big).\overrightarrow{n}_{2,\Gamma}$$

- The extended pressure condition (1^{st} form)

$$\eta_{\beta,1}(s_{1,\Gamma})(p_{\beta,1,\Gamma} - p_{\beta,2,\Gamma})^+ - \eta_{\beta,2}(s_{2,\Gamma})(p_{\beta,2,\Gamma} - p_{\beta,1,\Gamma})^+ = 0$$

Full Screen

- **The extended pressure condition (2^{nd} form)**



$$\eta_{\beta,1}(s_{1,\Gamma})(p_{\beta,1,\Gamma} - p_{\beta,2,\Gamma})^+ - \eta_{\beta,2}(s_{2,\Gamma})(p_{\beta,2,\Gamma} - p_{\beta,1,\Gamma})^+ = 0$$

$$\Updownarrow$$

$$\hat{\pi}_1(s_{1,\Gamma}) = \hat{\pi}_2(s_{2,\Gamma})$$

2. The mathematical model

The continuous problem

- A porous medium $\Omega = \Omega_1 \bigcup \Omega_2 \subset \mathbb{R}^d$
- A two-phase immiscible and incompressible flow only submitted to capillary effects
- For all $i \in \{1, 2\}$ on each domain Ω_i, we have

$$\phi(x, s) = \phi_i(s), \;\; \eta_\beta(x, s) = \eta_{\beta,i}(s), \;\; \pi(x, s) = \pi_i(s).$$

- The boundary $\partial\Omega$ is impermeable.

Full Screen

- For all $i \in \{1, 2\}$,

$$\phi_i \frac{\partial s}{\partial t} - \Delta \varphi_i(s) = 0 \text{ on } \Omega_i \times (0, T)$$

with $\varphi_i^{'}(s) = \dfrac{\eta_{o,i}\eta_{w,i}}{\eta_{o,i} + \eta_{w,i}}(s)\pi_i^{'}(s)$.

- On the interface, $\Gamma = \partial\Omega_1 \bigcap \partial\Omega_2$, we impose
  - the continuity of the flux :

$$\overrightarrow{\nabla}\varphi_1(s_{1,\Gamma}).\overrightarrow{n}_{1,2} = -\overrightarrow{\nabla}\varphi_2(s_{2,\Gamma}).\overrightarrow{n}_{2,1},$$

  - the extended pressure condition :

$$\hat{\pi}_1(s_{1,\Gamma}) = \hat{\pi}_2(s_{2,\Gamma}).$$

The weak problem

1. For all $i \in \{1,2\}$, $s = s_i$ on $\Omega_i \times (0,T)$ with $s_i \in L^\infty(\Omega_i \times (0,T))$, $0 \le s_i \le 1$, $\varphi_i(s_i) \in L^2((0,T), H^1(\Omega_i))$,

2. $$\int_0^T \int_{\Omega_1} s_1 \psi_t - \overrightarrow{\nabla} \varphi_1(s_1).\overrightarrow{\nabla} \psi \; dxdt + \int_0^T \int_{\Omega_2} s_2 \psi_t - \overrightarrow{\nabla} \varphi_2(s_2).\overrightarrow{\nabla} \psi \; dxdt + \int_\Omega s_{ini} \psi(.,0) dx = 0,$$

   $C_{test} = \{\psi \in H^1(\Omega \times (0,T)) \;/\; \psi(.,T) = 0\}$.

3. The function $w = \Psi(\hat{\pi}_i(s_i))$ on $\Omega_i \times (0,T)$ with

$$\Psi : \begin{cases} [\pi_2(0), \pi_1(1)] \longrightarrow \mathbb{R} \\ p \longmapsto \displaystyle\int_{\pi_2(0)}^p \min(\lambda_1(\pi_1^{(-1)}(a)), \lambda_2(\pi_2^{(-1)}(a)))da \end{cases}$$

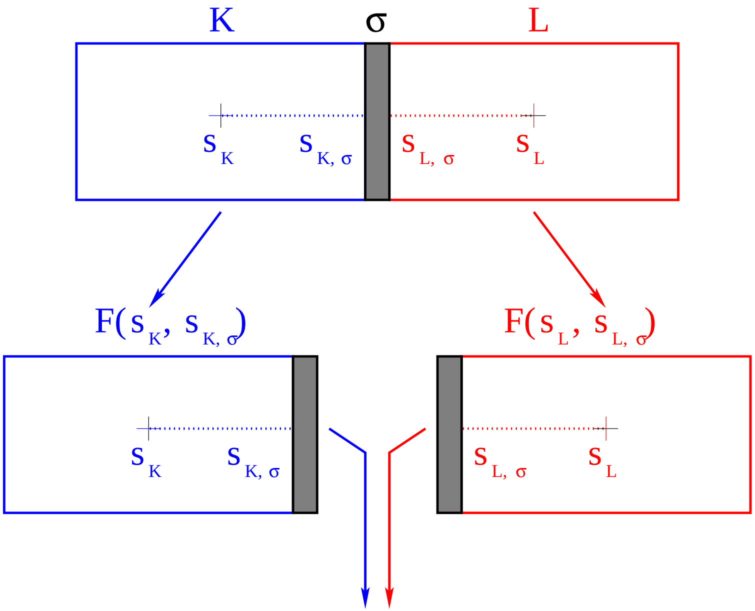
   belongs to $L^2((0,T), H^1(\Omega))$.

Capillary trapping
Mathematical Model
A FV scheme
Numerical Tests
Concluding Remarks

Full Screen

3. A FV scheme

Principle



Elimination of $s_{K,\sigma}$ and $s_{L,\sigma}$ thanks to

_ the flux conservation

_ the extended pressure condition

Definition

For all $n \in \{0, \ldots, M\}$, $(s_K^{n+1})_{K \in \mathcal{T}}$ satisfies for all $K \in \mathcal{T}_i$, $i \in \{1, 2\}$,

$$m(K)\frac{s_K^{n+1} - s_K^n}{\delta t} + \sum_{L \in N(K), L \in \mathcal{T}_i} \tau_{K|L} \left(\varphi_i(s_K^{n+1}) - \varphi_i(s_L^{n+1})\right) + \sum_{\sigma \in \mathcal{E}_{int,1,2} \bigcap \mathcal{E}_K} \frac{m(\sigma)}{d_{K,\sigma}} \left(\varphi_i(s_K^{n+1}) - \varphi_i(s_{K,\sigma}^{n+1})\right) = 0$$

where, for all $K|L \in \mathcal{E}_{int,1,2}$, $K \in \mathcal{T}_1$ and $L \in \mathcal{T}_2$, $s_{K,K|L}^{n+1}$ and $s_{L,K|L}^{n+1}$ are solutions to

$$\left\{ \begin{array}{l} \dfrac{\varphi_1(s_K^{n+1}) - \varphi_1(s_{K,K|L}^{n+1})}{d_{K,K|L}} = \dfrac{\varphi_2(s_{L,K|L}^{n+1}) - \varphi_2(s_L^{n+1})}{d_{L,K|L}}, \\ \hat{\pi}_1(s_{K,K|L}^{n+1}) = \hat{\pi}_2(s_{L,K|L}^{n+1}). \end{array} \right.$$

Full Screen

Properties of the scheme

- The scheme admits a unique solution.
- For all $n \in \{0, \ldots, M\}$, $s_{\mathcal{M}}^{n+1}$ satisfies

$$\forall K \in \mathcal{T}, 0 \leq s_K^{n+1} \leq 1.$$

- Up to a subsequence, $(s_{\mathcal{D}_m})_{m \in \mathbb{N}}$ is such that

$$s_{\mathcal{D}_m} \xrightarrow{m \to +\infty} s_1 \text{ in } L^q(\Omega_1 \times (0,T)),$$
$$s_{\mathcal{D}_m} \xrightarrow{m \to +\infty} s_2 \text{ in } L^q(\Omega_2 \times (0,T))$$

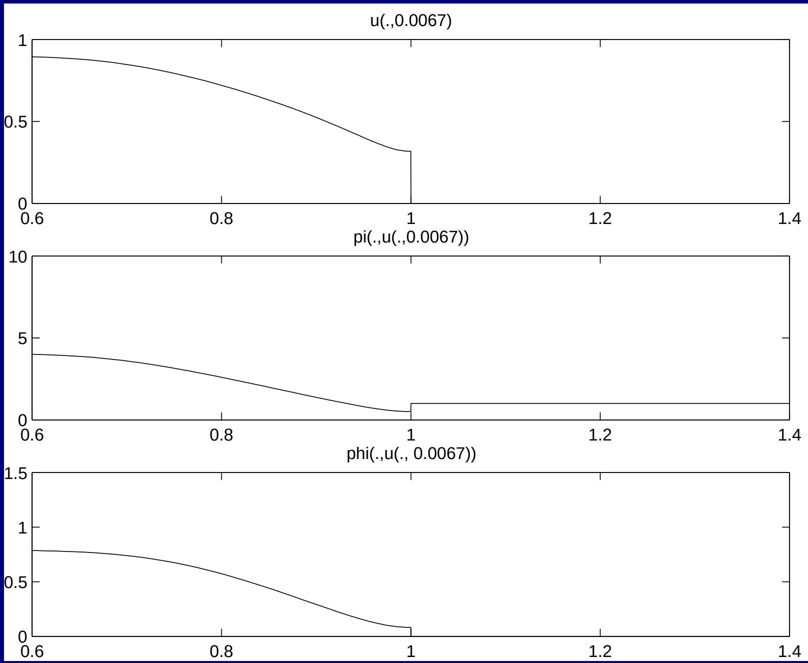
where $1 \leq q < \infty$ and for all $i \in \{1,2\}$, $s_i \in L^\infty(\Omega_i \times (0,T))$ and $\varphi_i(s_i) \in L^2((0,T), H^1(\Omega_i))$. The functions s_i, $i \in \{1,2\}$, satisfy the weak problem.

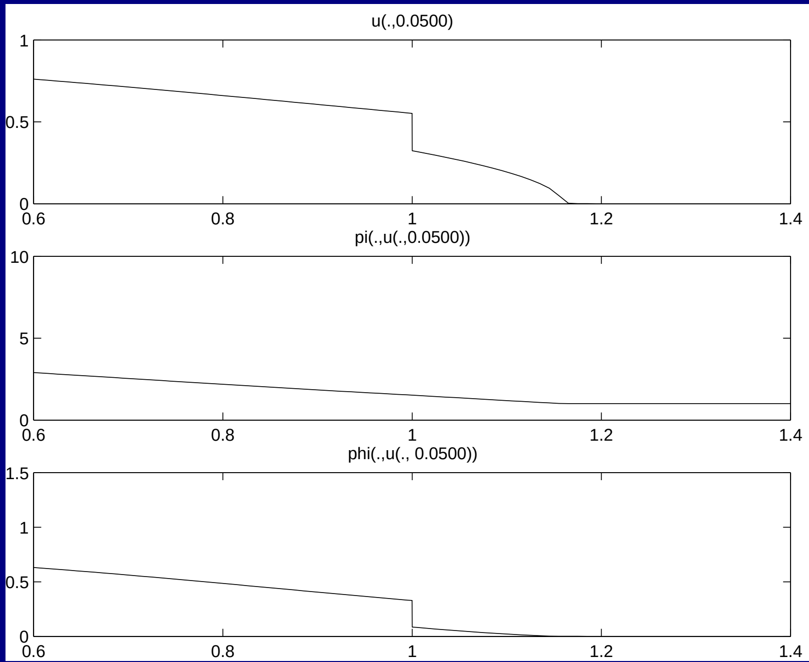
Full Screen

4. Numerical tests

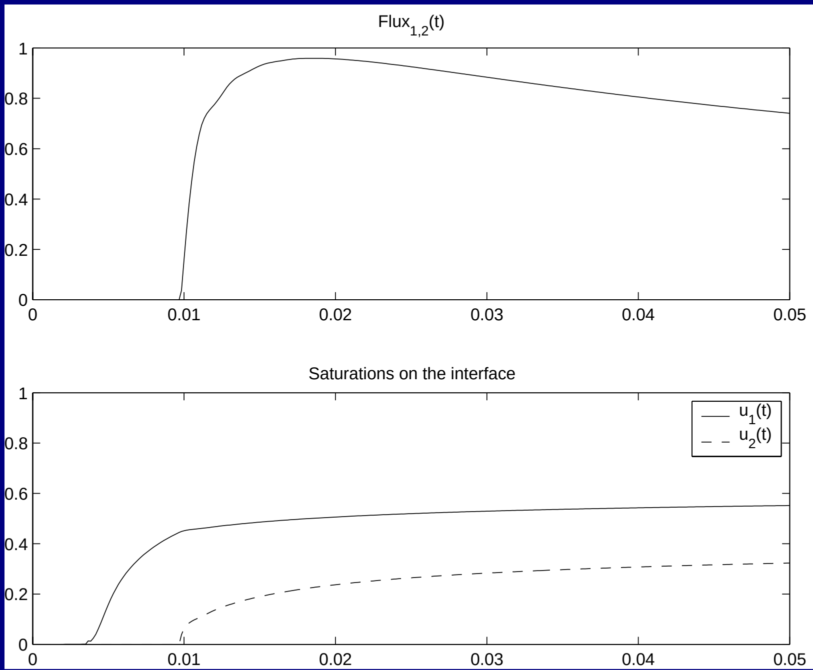
Test1

- $\Omega_1 =]0,1[$, $\Omega_2 =]1,2[$, $\phi_1 = \phi_2 = 1$
- $\eta_o(s) = s$, $\eta_w(s) = 1 - s$, $\pi_1(s) = 5s^2$, $\pi_2(s) = 5s^2 + 1$
- $s_{\text{ini}}(x) = \begin{cases} 0.9 & \text{if } x < 0.9 \\ 0 & \text{otherwise} \end{cases}$
- $\delta t = \frac{1}{6}.10^{-3}$, $\delta x = 10^{-2}$


u(.,0.0067)
pi(.,u(.,0.0067))
phi(.,u(., 0.0067))
0.6
0.8
1
1.2
1.4


u(.,0.0500)
1
0.5
0
0.6
0.8
1
1.2
1.4
pi(.,u(.,0.0500))
10
5
0
0.6
0.8
1
1.2
1.4
phi(.,u(., 0.0500))
1.5
1
0.5
0
0.6
0.8
1
1.2
1.4

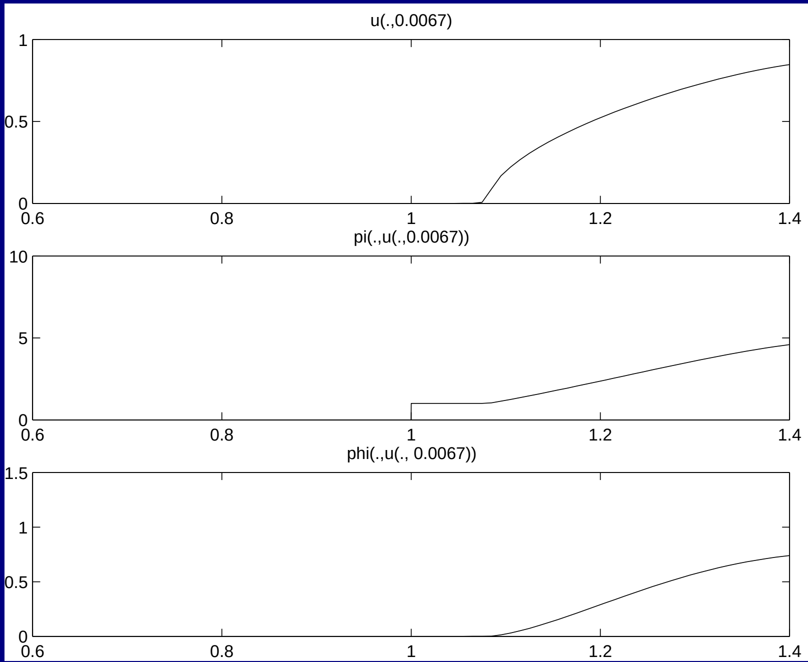
Full Screen

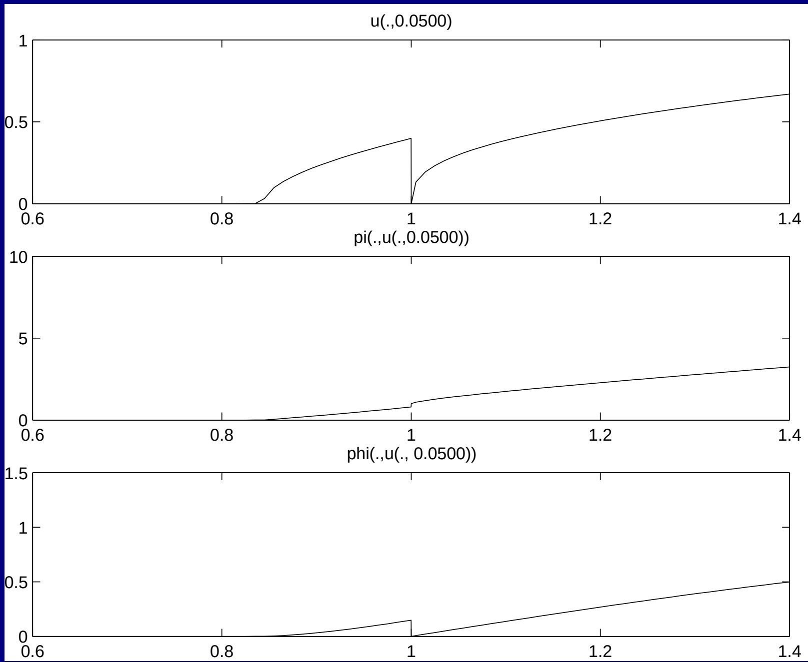

Flux$_{1,2}$(t)
1
0.8
0.6
0.4
0.2
0
0
0.01
0.02
0.03
0.04
0.05
Saturations on the interface
u$_1$(t)
u$_2$(t)

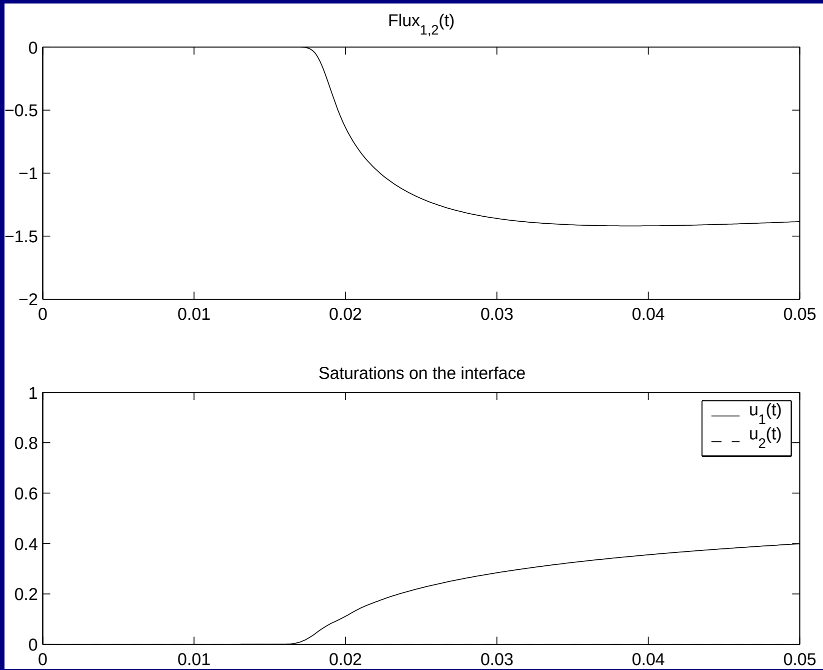
Full Screen

Test2

- $\Omega_1 =]0,1[$, $\Omega_2 =]1,2[$, $\phi_1 = \phi_2 = 1$
- $\eta_o(s) = s$, $\eta_w(s) = 1 - s$, $\pi_1(s) = 5s^2$, $\pi_2(s) = 5s^2 + 1$
- $s_{\text{ini}}(x) = \begin{cases} 0.9 & \text{if } x > 1.2 \\ 0 & \text{otherwise} \end{cases}$
- $\delta t = \frac{1}{6}.10^{-3}$, $\delta x = 10^{-2}$


u(.,0.0067)
1
0.5
0
0.6
0.8
1
1.2
1.4
pi(.,u(.,0.0067))
10
5
0
0.6
0.8
1
1.2
1.4
phi(.,u(., 0.0067))
1.5
1
0.5
0
0.6
0.8
1
1.2
1.4


u(.,0.0500)
1
0.5
0
0.6
0.8
1
1.2
1.4
pi(.,u(.,0.0500))
10
5
0
0.6
0.8
1
1.2
1.4
phi(.,u(., 0.0500))
1.5
1
0.5
0
0.6
0.8
1
1.2
1.4


Flux1,2(t)
0
−0.5
−1
−1.5
−2
0
0.01
0.02
0.03
0.04
0.05
Saturations on the interface
1
0.8
0.6
0.4
0.2
0
u1(t)
u2(t)

5. Concluding Remarks

- A better understanding of the capillary trapping
- Conditions to improve the accuracy of the numerical schemes
- These results can be extended to a model taking the gravity and a total throughput into account.
- It remains to prove the uniqueness of a solution satisfying the weak problem...

Full Screen

References

[1] M. Bertsch, R. D. Passo, and C. van Duijn, Analysis of oil trapping in porous media flow, SIAM Journal on Mathematical Analysis, 35 (2003), pp. 245–267.

[2] C. van Duijn, J. Molenaar, and M. de Neef, The effect of capillary forces on immiscible two-phase flow in heterogeneous porous media, Transport in Porous Media, (1995).

[3] G. Enchéry, R. Eymard, and A. Michel, Numerical approximation of a two-phase flow problem in a porous medium with discontinuous capillary forces, soumis.